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A REMARK ON THE SIZE OF $\pi_q(S^n)$

Carl-Friedrich Bödigheimer and Hans-Werner Henn

Classical results of homotopy theory are used to prove that $\dim \pi_q(S^n)\otimes \mathbb{Z}_p \leq 3^{(q-\frac{n}{2})}$ for all primes p. Simultaneously we obtain bounds on the order of the p-torsion subgroup of $\pi_q(S^n)$.

Introduction. It might be known to many algebraic topologists that the work of James [J_2] and Toda [T] on the (double) suspension provide tools to get bounds on the rank of $\pi_q(S^n)$ and its torsion subgroup. Recently Selick [S] has given a rather crude estimate for the p-rank of $\pi_q(S^n)$ for odd primes only. In a more direct approach than he we offer better bounds for the p-rank of $\pi_q(S^n)$ and the order of its p-torsion subgroup for all primes based on the simple idea that the size of the middle group of an exact sequence is bounded by the size of the two outer groups.

Let p be a prime and denote by $O_p(q,n)$ the order of the torsion subgroup of $\pi_q(S^n_{(p)})$ and by $R_p(q,n)$ the rank of

$\pi_q(S^n_{(p)})$. For p odd we always concentrate on odd spheres because of Serre's splitting $\Omega S^{2n}_{(p)} \simeq S^{2n-1}_{(p)} \times \Omega S^{4n-1}_{(p)}$ in [S_2]. We first prove some Fibonacci type recursion formulae.

Lemma. (i) If $p = 2$, then

(1) $R_2(q,n+1) \doteq R_2(q-1,n)+R_2(q,2n+1)$

(2a) $O_2(q,n+1) \doteq 2\cdot O_2(q-1,n)$ for $(q,n)=(4,2),(8,4),(16,8)$

(2b) $O_2(q,n+1) \doteq O_2(q-1,n)\cdot O_2(q,2n+1)$ otherwise.

(ii) If $p > 2$, then

(3) $R_p(q,2n+1) \doteq R_p(q-2,2n-1)+R_p(q-1,2np-1)+R_p(q,2np+1)$

(4a) $O_p(q,2n+1) \doteq p\cdot O_p(q-2,2n-1)$ for $q=2np$, otherwise

(4b) $O_p(q,2n+1) \doteq O_p(q-2,2n-1)\cdot O_p(q-1,2np-1)\cdot O_p(q,2np+1)$.

Proof. (i) For every $n \geq 1$ there is a 2-local fibration [J_2]

$$S^n \xrightarrow{E} \Omega S^{n+1} \longrightarrow \Omega S^{2n+1}$$

where E is the suspension map. Applying π_{q-1} gives (1). Also (2b) follows provided all groups involved are finite. Recall that $\pi_q(S^k)$ is torsion and thus finite unless $q=k$, or $q=2k-1$ and k is even [S_1]; that means not all of our groups are finite if $q=n+1$, or $q=2n+1$, or $q=2n$ and n even. Obviously (2b) holds in the first two cases. If $q=2n$ and n is even then E_* is epic and its kernel is generated by the Whitehead product $[id_n, id_n]$ of the identity of S^n which has Hopf invariant 2 [W;p.495,549]. Therefore $\ker E_*$ is infinite cyclic and a direct summand unless $\pi_{2n-1}(S^n)$ contains an element with Hopf invariant 1, i.e. unless $n=2,4$ or 8 [A].

Hence (2a) and (2b) follow easily.

(ii) Toda's two p-local fibrations [T]

$$J_{p-1}S^{2n} \longrightarrow \Omega S^{2n+1} \longrightarrow \Omega S^{2np+1}$$

$$S^{2n-1} \longrightarrow \Omega J_{p-1}S^{2n} \longrightarrow \Omega S^{2np-1}$$

where J_k is the k-th stage of the James construction [J_1], imply (3) and (4b) since now the only non-trivial case with infinite groups occurs for q=2np; now (4a) follows from [T] or [H;p.3o9f].

Denote by $S_p(q,n)$ either $R_p(q,n)$ or $\log_p O_p(q,n)$ where again n is odd if $p > 2$. Our bound reads as follows.

<u>Proposition</u>. $S_p(q,n) \leq 3^{q-\frac{n}{2}}$ <u>for</u> <u>all</u> <u>primes</u>.

<u>Furthermore</u>, <u>if</u> $0 < \sigma < 1$ <u>then</u> $S_p(q,n) \leq 3^{\sigma(q-\frac{n}{2})}$ <u>for</u> <u>almost</u> <u>all</u> <u>primes</u>.

Proof: For each prime the inequalities hold obviously (for any positive value of σ) if n=1 or $n \geq q$. Suppose now they hold for all (q',n') with $q' < q$ or q'=q and $n+1 < n'$. By applying the Lemma (twice if p=2) we arrive at

$$S_p(q,n+1) \leq 3^{\sigma\,(q-2-\frac{n-1}{2})} + 3^{\sigma\,(q-1-\frac{np-1}{2})} + 3^{\sigma\,(q-\frac{np+1}{2})}$$

$$= 3^{\sigma\,(q-\frac{n+1}{2})} \left[3^{-\sigma} + 2\cdot 3^{-\sigma\,\frac{n(p-1)}{2}} \right] .$$

For $n \geq 2$ this yields

$$S_p(q,n+1) \leq 3^{\sigma(q-\frac{n+1}{2})}\left[3^{-\sigma} + 2\cdot 3^{-\sigma(p-1)}\right]$$
$$\leq 3^{\sigma(q-\frac{n+1}{2})}$$

provided we have $\sigma = 1$ or p large enough. The case n=1 remains for the prime 2 only; we find for $q > 2$
$S_2(q,2) \leq S_2(q,3) + S_2(q-1,1) \leq 3^{\sigma(q-\frac{3}{2})} \leq 3^{\sigma(q-1)}$
using the Lemma and $S_2(q-1,1) = 0$.

Remarks. a) The proof actually shows that the second inequality holds for $p \geq 1 - \frac{1}{\sigma}\log_3(\frac{1-3^{-\sigma}}{2})$.

b) There are sequences b(q,n) which satisfy (1) or (3), respectively, and grow exponentially (for fixed p and n) in the sense that for some $d > 0$ and $C > 1$ we have $b(q,n) \geq d\cdot C^q$ for almost all q. Hence to produce smaller than exponential bounds on the growth of $S_p(q,n)$ one needs more information from homotopy theory than provided by the Lemma.

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C.-F.Bödigheimer, H.-W.Henn*

Mathematisches Institut der Universität

Im Neuenheimer Feld 288

D - 69oo Heidelberg

West Germany

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