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ADAPTIVE CONVEXIFICATION FOR ROBUST OPTIMIZATION PROBLEMS

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Survey

1. Introduction
2. The role of lower level global optimality
3. Three approaches to treat the lower level
4. A fourth approach: adaptive convexification
5. Numerical results
6. Future directions

Robust optimization

Given: a parametric finite optimization problem

$$FP(p): \quad \min_x f(x,p) \quad \text{s.t.} \quad g(x,p) \le 0$$

with uncertain parameter p in compact uncertainty set P (without stochastic information).

Robust optimization

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Robust (pessimistic, worst-case) **formulation:**

$$RP: \quad \min_x \max_{p \in P} f(x,p) \quad \text{s.t.} \quad g(x,p) \le 0 \quad \forall\, p \in P$$

Example: Portfolio optimization
(Ben-Tal/Nemirovski 1999)

Invest 1 € in a portfolio comprised of N shares.

Return of share i after one year: p_i .

Wanted: optimal portfolio structure, i.e. amount x_i to be invested in share i , such that portfolio value is maximized:

$$\max_x \; p^\top x \quad \text{s.t.} \quad \sum_{i=1}^{N} x_i = 1\,,\; x \geq 0\;.$$

Example: Portfolio optimization
(Ben-Tal/Nemirovski 1999)

Problem: $p_i\,,\;\, i = 1, ..., N\,,$ unknown.

Pessimistic approach: $p \in P$ (nonempty, compact)

$$\max_x \; \min_{p \in P} \; p^\top x \quad \text{s.t.} \quad \sum_{i=1}^{N} x_i = 1\,,\;\, x \geq 0\;.$$

Formulation with max-constraints

$$RP: \quad \min_x \max_{p \in P} f(x,p) \quad \text{s.t.} \quad g(x,p) \leq 0 \quad \forall\ p \in P$$

For continuous g and nonempty compact P:

$$g(x,p) \ \leq \ 0 \quad \forall\ p \in P \qquad \Leftrightarrow \qquad \max_{p \in P} g(x,p) \ \leq \ 0$$

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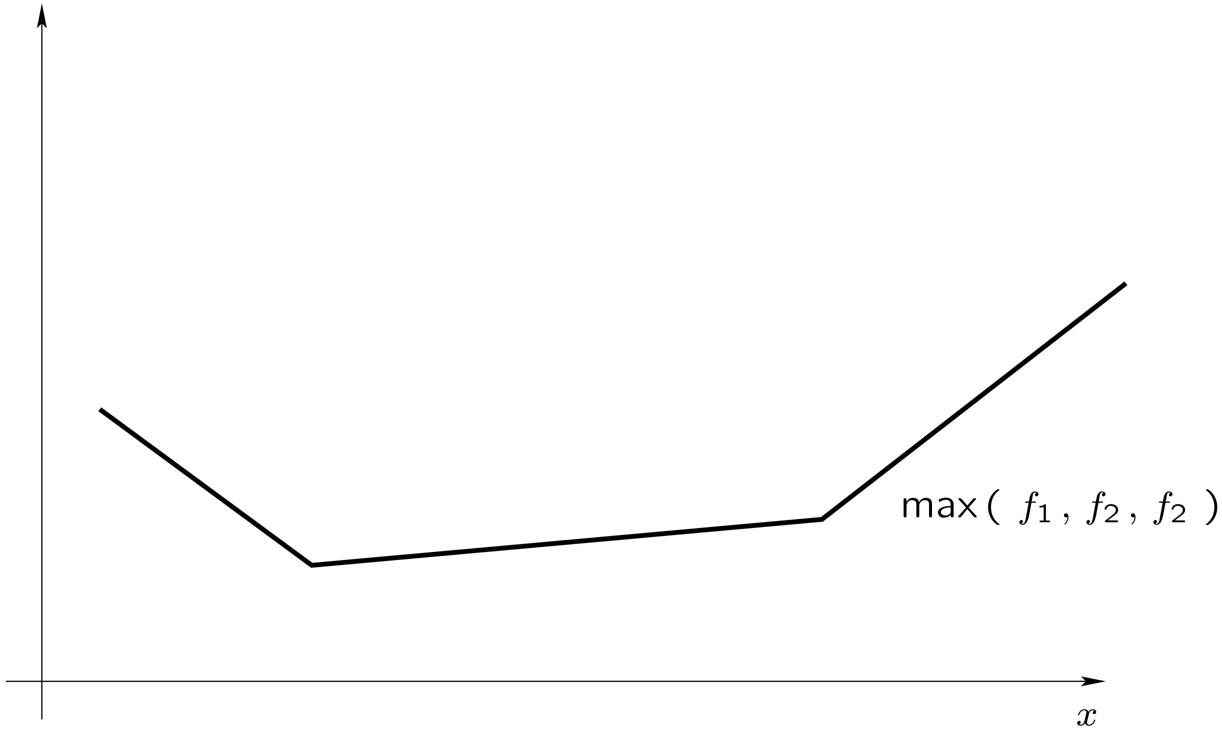
For continuous g and nonempty compact P:

$$g(x,p) \;\le\; 0 \quad \forall\, p \in P \qquad \Leftrightarrow \qquad \max_{p \in P} g(x,p) \;\le\; 0$$

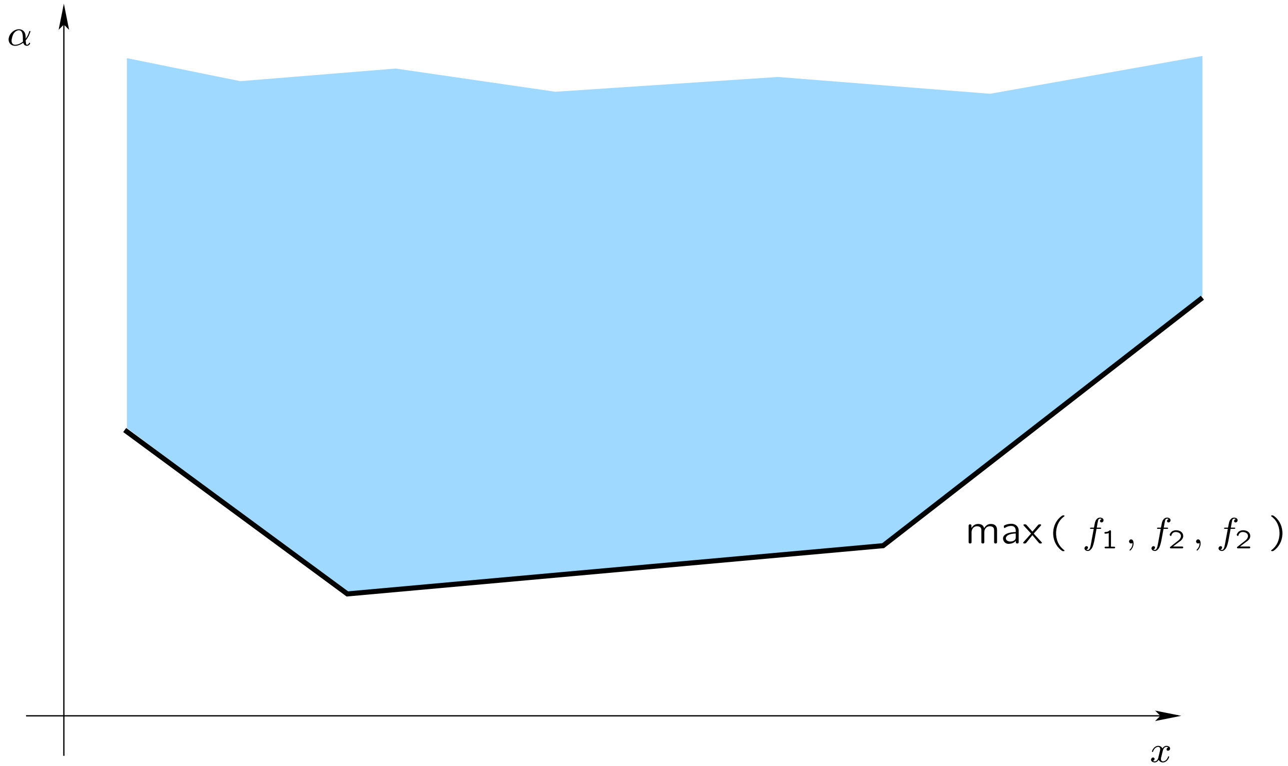
and (by epigraph reformulation)

$$\min_x \max_{p \in P} f(x,p) \quad = \quad \min_{x,\alpha} \alpha \quad \text{s.t.} \quad \max_{p \in P} f(x,p) \;\le\; \alpha$$

Epigraph reformulation



Epigraph reformulation



Formulation with max-constraints

$$RP: \quad \min_{x} \max_{p \in P} f(x,p) \quad \text{s.t.} \quad g(x,p) \leq 0 \quad \forall\; p \in P$$

is hence equivalent to

$$\begin{aligned} \min_{x,\alpha}\ \alpha \quad \text{s.t.} \quad \max_{p \in P} f(x,p) \ &\leq\ \alpha \\ \max_{p \in P} g(x,p) \ &\leq\ 0 \end{aligned}$$

Main problem: global optimality

To check the feasiblity of x, compute the optimal value $\varphi(x)$ of the problem

$$Q(x): \qquad \max g(x,p) \quad \text{s.t.} \quad p \in P$$

and verify $\varphi(x) \leq 0$.

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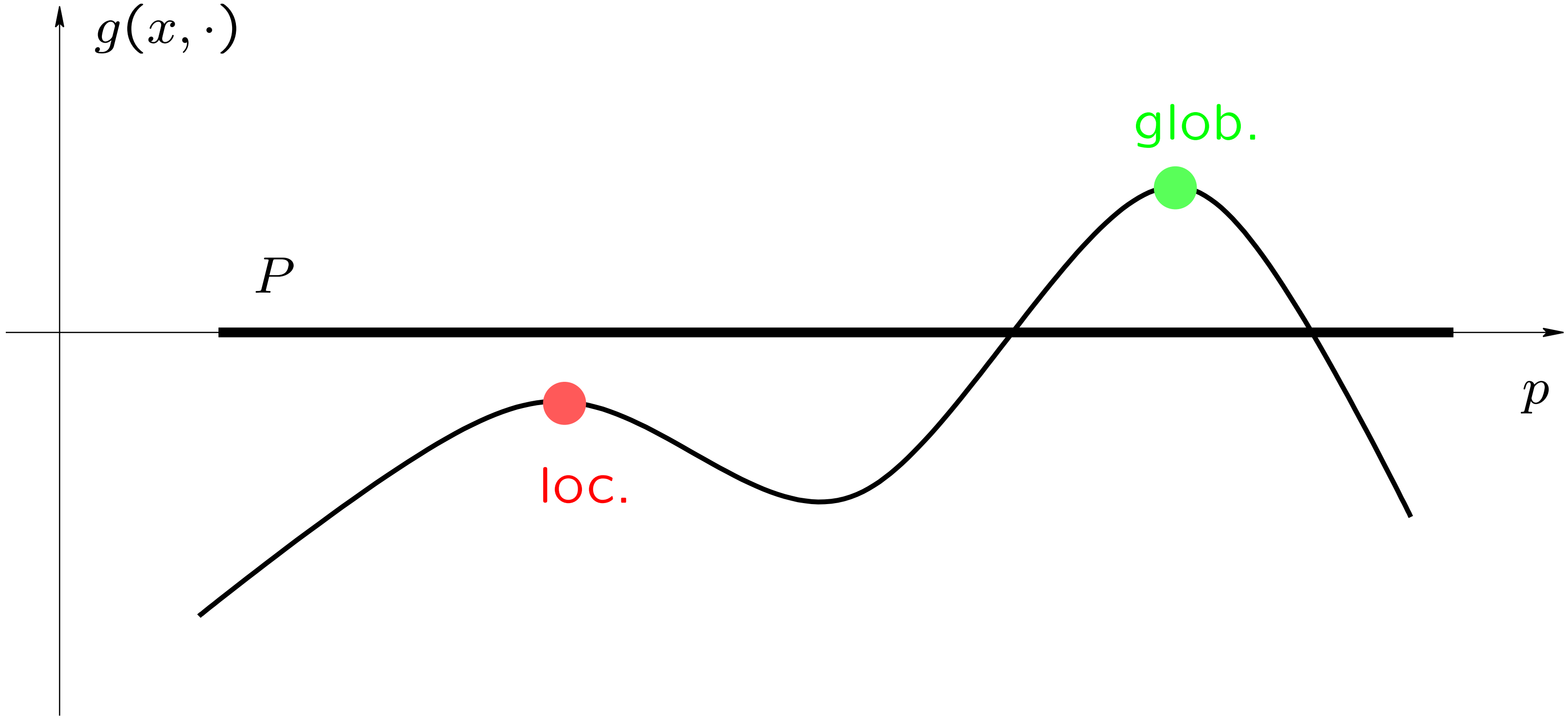
but this is not sufficient to check feasibility !

Local vs. global optimality



Remedies

1. Make problem 'tractable' by strong assumptions that ensure **explicit formulas** for $\varphi(x)$ (Ben-Tal/Nemirovski, iterates are interior points for RP)

2. Only assume convexity of the lower level problem ($RP \Leftrightarrow MPCC$, but no applications known)

3. Use classical semi-infinite optimization routines (implementation issues, iterates are usually infeasible for RP)

4. Use the adaptive convexification algorithm (**weak assumptions and feasible iterates for RP**).

Remedy 1: A special case – ellipsoidial uncertainty

Ben-Tal/Nemirovski (1999) assume $g(x, \cdot)$ **linear**:

$$g(x, p) \;=\; a(x)^\top p - b(x)$$

and P **ellipsoidial**:

$$P \;=\; \{\; p | \; p^\top H p + r^\top p + s \leq 0 \;\} \qquad (\, H \succ 0 \,)$$

Remedy 1: A special case – ellipsoidial uncertainty

Effect:

$\varphi(x) \ = \ \max_{p \in P} g(x,p)$ becomes a **nice** nonsmooth function.

e.g. for $g(x,p) \ = \ a(x)^\top p - b(x)$ and $P \ = \ \{ \ p | \ p^\top p \leq 1 \ \}$

the global maximizer of $Q(x)$ is $p^\star \ = \ a(x)/||a(x)||_2$,

and due to

$$\varphi(x) \ = \ a(x)^\top p^\star - b(x) \ = \ ||a(x)||_2 - b(x)$$

x is feasible if and only if $||a(x)||_2 \ \leq \ b(x)$.

Remedy 1: A very special case

Using Schur complements one shows

$$||a(x)||_2 \;\leq\; b(x) \quad \Leftrightarrow \quad \begin{pmatrix} b(x)\cdot I & a(x) \\ a(x)^\top & b(x) \end{pmatrix} \succeq 0 \; .$$

Hence, if **moreover** a and b are linear in x, the feasibility problem is equivalent to a linear matrix inequality.

RP can then be treated by polynomial time algorithms (see Ben-Tal/El Ghaoui/Nemirovski 2000).

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Under similarly strong assumptions:
Saddle point approach by Tütüncü/Koenig 2004.

Remedy 2: RP with convex lower level solvable as MPCC

We only consider a lower level problem

$$Q(x): \qquad \max g(x,p) \quad \text{s.t.} \quad p \in P$$

with

$$P \;=\; [0,1] \;=\; \{\, p \in \mathbb{R} |\; p \geq 0,\; 1-p \geq 0 \,\}.$$

$Q(x)$ is assumed to be convex, that is,
$g(x,\cdot)$ is concave for all $x \in X$.

In the sequel we will need the Lagrangian of $Q(x)$,

$$\mathcal{L}(x,p,\gamma_\ell,\gamma_u) \;=\; g(x,p) + \gamma_\ell\, p + \gamma_u (1-p),$$

which obviously satisfies

$$\nabla_p \mathcal{L}(x,p,\gamma_\ell,\gamma_u) \;=\; \nabla_p g(x,p) + \gamma_\ell - \gamma_u\,.$$

Remedy 2: RP with convex lower level solvable as MPCC

$$\boldsymbol{RP}: \qquad \min_x \ f(x) \qquad \text{s.t.} \quad \max_{p \in P} \ g(x,p) \ \leq \ 0$$

$\Updownarrow$ (St./Still '00)

$$\boldsymbol{SG}: \qquad \min_{x,p} \ f(x) \qquad \text{s.t.} \quad g(x,p) \ \leq \ 0 \ ,$$

$$p \text{ solves } Q(x): \ \max_p g(x,p) \ \text{ s.t. } \ p \in P$$

$\Updownarrow$ $Q(x)$ convex, regular

$$\boldsymbol{MPCC}: \quad \min_{x,p,\gamma_\ell,\gamma_u} \ f(x) \quad \text{s.t.} \quad \begin{aligned} g(x,p) \quad &\leq \quad 0 \\ \nabla_p g(x,p) + \gamma_\ell - \gamma_u \quad &= \quad 0 \\ \gamma_\ell \geq 0, \ p \geq 0, \ \gamma_\ell \cdot p \quad &= \quad 0 \\ \gamma_u \geq 0, \ 1 - p \geq 0, \ \gamma_u \cdot (1-p) \quad &= \quad 0 \end{aligned}$$

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$$\boldsymbol{SG}: \qquad \min_{x,p} \; f(x) \qquad \text{s.t.} \quad g(x,p) \;\le\; 0 \;,$$

$$\boldsymbol{p} \text{ **solves** } \boldsymbol{Q(x)}\boldsymbol{:} \; \max_{\boldsymbol{p}} \boldsymbol{g(x,p)} \; \text{**s.t.**} \; \boldsymbol{p \in P}$$

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$$\boldsymbol{MPCC}: \quad \min_{x,p,\gamma_\ell,\gamma_u} \; f(x) \quad \text{s.t.} \quad \begin{array}{rcl} g(x,p) & \le & 0 \\ \boldsymbol{\nabla_p g(x,p) + \gamma_\ell - \gamma_u} & \boldsymbol{=} & \boldsymbol{0} \\ \boldsymbol{\gamma_\ell \ge 0,\; p \ge 0,\; \gamma_\ell \cdot p} & \boldsymbol{=} & \boldsymbol{0} \\ \boldsymbol{\gamma_u \ge 0,\; 1-p \ge 0,\; \gamma_u \cdot (1-p)} & \boldsymbol{=} & \boldsymbol{0} \end{array}$$

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$$\boldsymbol{MPCC}: \quad \min_{x,p,\gamma_\ell,\gamma_u} \; f(x) \quad \text{s.t.} \quad \begin{array}{rcl} g(x,p) & \le & 0 \\ \nabla_p g(x,p) + \gamma_\ell - \gamma_u & = & 0 \\ \boldsymbol{\gamma_\ell \ge 0, \; p \ge 0, \; \gamma_\ell \cdot p} & = & \mathbf{0} \\ \boldsymbol{\gamma_u \ge 0, \; 1 - p \ge 0, \; \gamma_u \cdot (1-p)} & = & \mathbf{0} \end{array}$$

Remedy 2: RP with convex lower level solvable as MPCC

$$\boldsymbol{RP}: \qquad \min_x \ f(x) \qquad \text{s.t.} \quad \max_{p \in P} \ g(x,p) \ \leq \ 0$$

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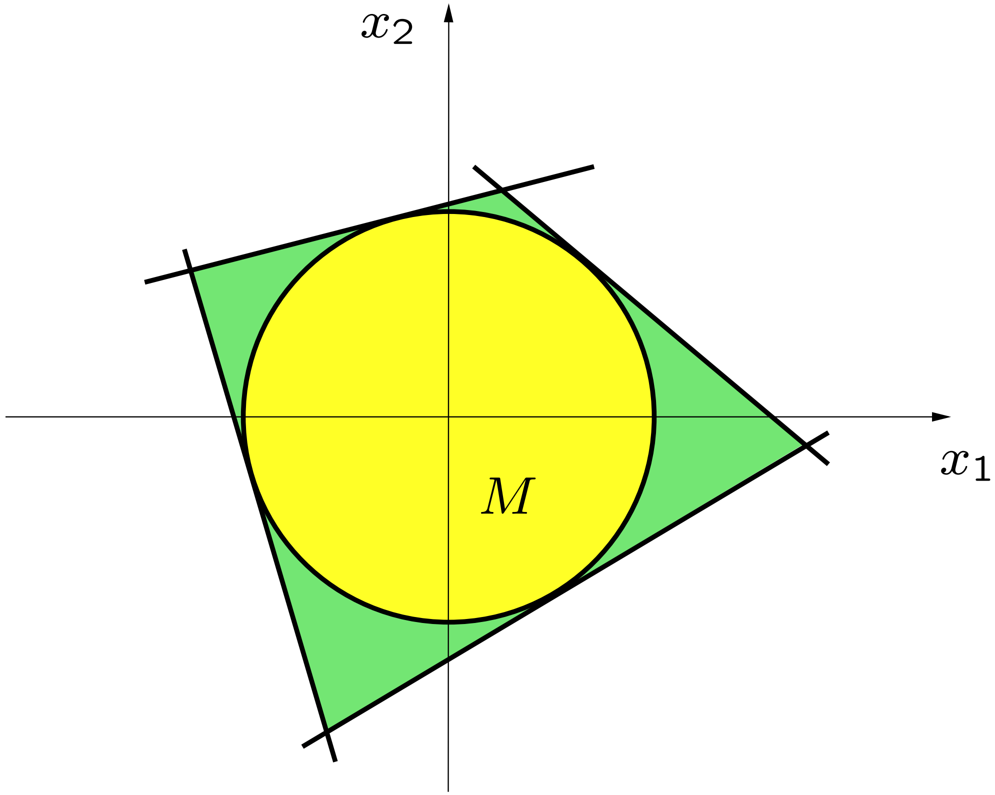
$$p \text{ solves } Q(x): \ \max_p \ g(x,p) \ \text{ s.t. } \ p \in P$$

$\Updownarrow$ $Q(x)$ convex, regular

$$\boldsymbol{FP^\star}: \qquad \min_{x,p,\gamma_\ell,\gamma_u} \ f(x) \quad \text{s.t.} \quad \begin{aligned} g(x,p) \quad &\leq \quad 0 \\ \nabla_p g(x,p) + \gamma_\ell - \gamma_u \quad &= \quad 0 \\ \boldsymbol{\Phi\left(\gamma_\ell, p\right)} \quad &\boldsymbol{=} \quad \boldsymbol{0} \\ \boldsymbol{\Phi\left(\gamma_u, 1-p\right)} \quad &\boldsymbol{=} \quad \boldsymbol{0} \end{aligned}$$

Remedy 3: Idea of discretization methods



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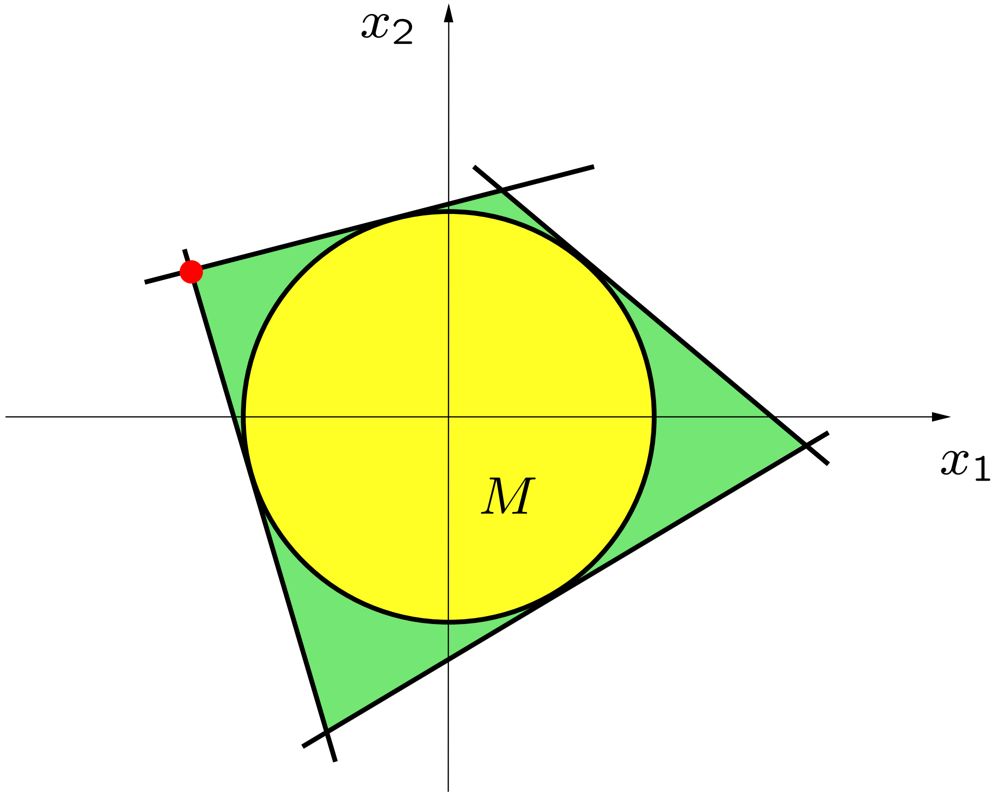

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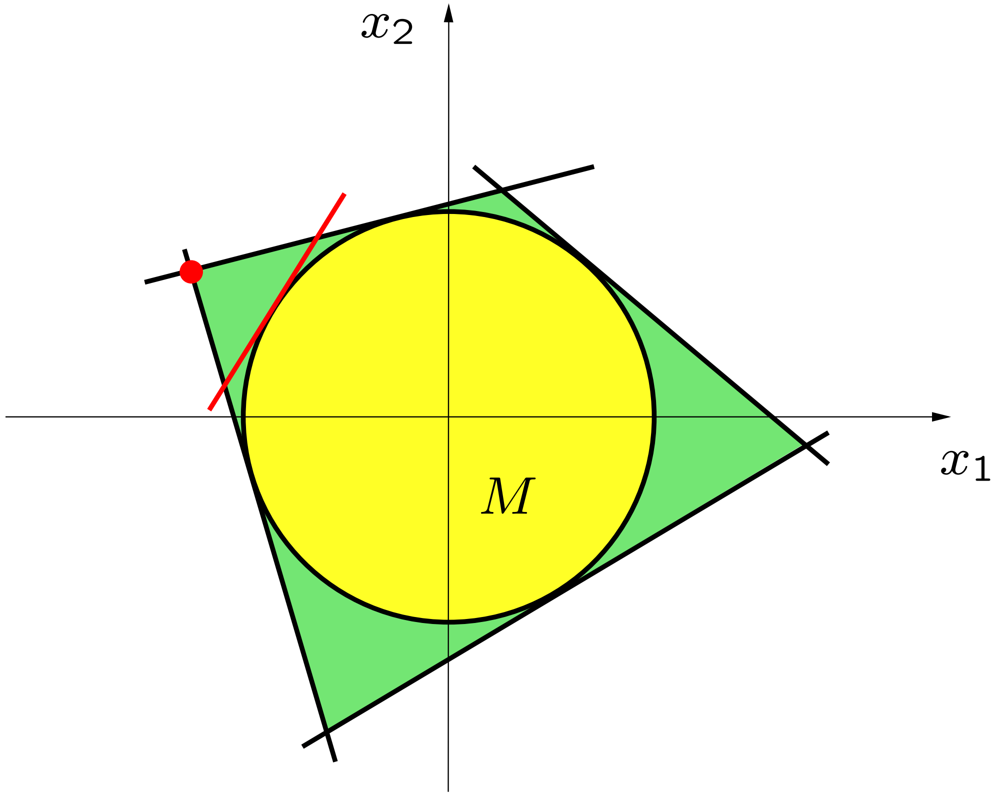

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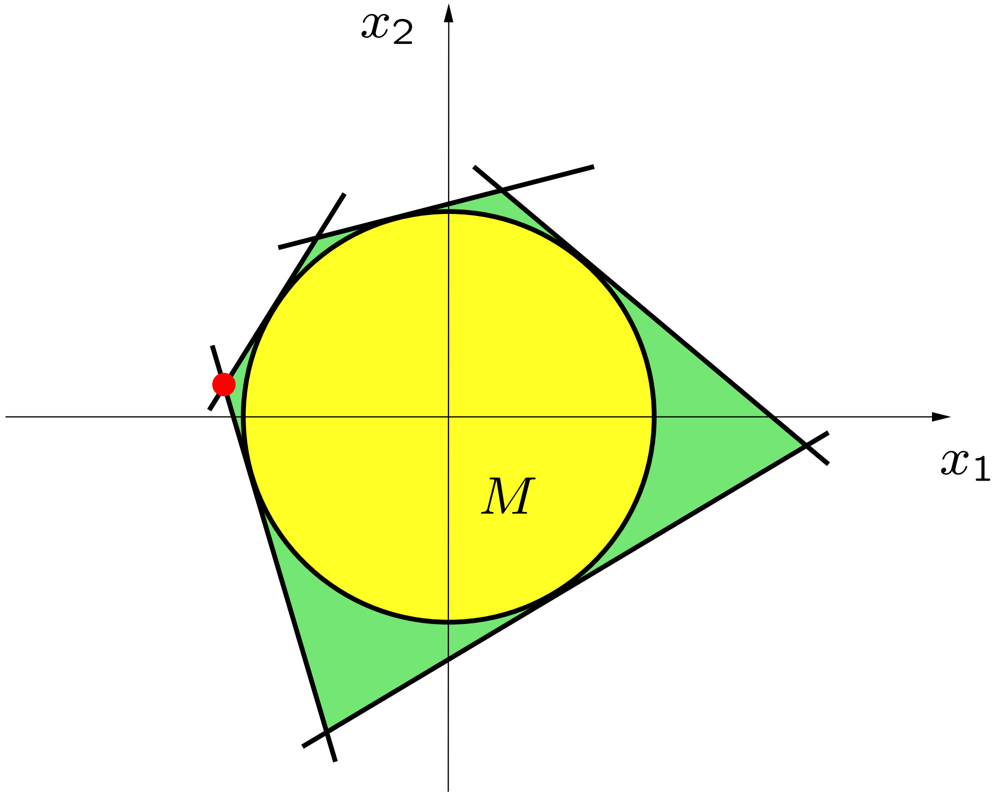


Remedy 4: The adaptive convexification algorithm

- Needs no convexity assumption on the lower level problem.
- All iterates are feasible for RP.
- Even broader scope than robust optimization.
- Based on interval arithmetic.
- Coauthor: Chris Floudas, Princeton University.

Semi-infinite Optimization

$SIP:$ minimize $f(x)$ over $X \cap M$

with

$$X = [x^\ell, x^u] \subset \mathbb{R}^n,$$

$$M = \{\ x \in \mathbb{R}^n \mid g(x,y) \leq 0 \ \text{ for all } \ y \in Y\ \},$$

$$Y = [y^\ell, y^u] \subset \mathbb{R}^m,$$

and with C^2 functions

$$f: \ \mathbb{R}^n \to \mathbb{R}\,,$$

$$g: \ \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p.$$

Applications

- Robust Optimization
- Chebyshev Approximation
- Minimax problems
- Design Centering
- Defect minimization for boundary value problems

$\vdots$

αBB for SIP with nonconvex lower level

If $Q(x)$ it not convex, that is,

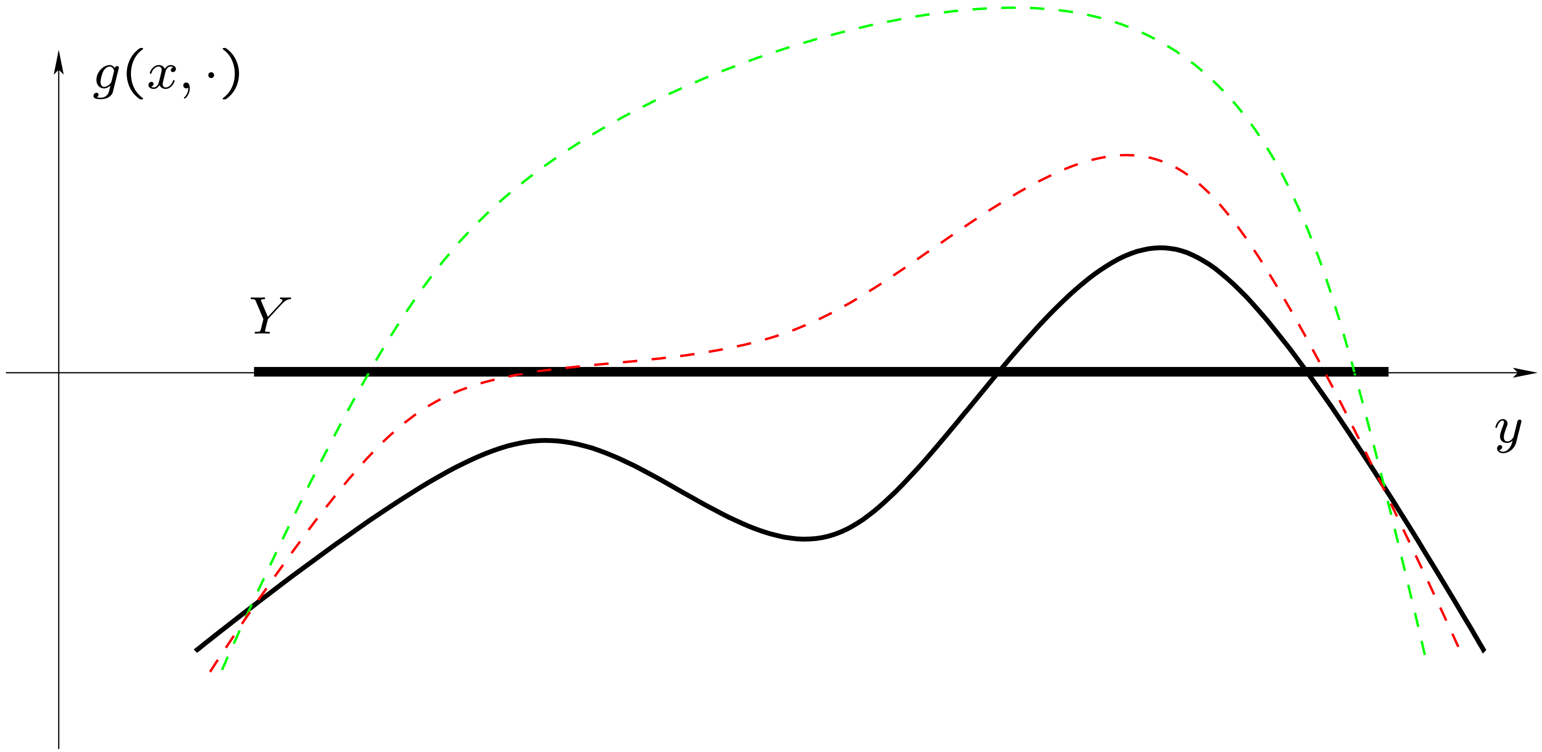
if $g(x, \cdot)$ is not concave for all $x \in X$,

by the αBB method (Adjiman/Androulakis/Floudas 1998)

we construct *concave overestimators* for $g(x, \cdot)$:

$$\tilde{g}(x, y; \alpha) \;=\; g(x, y) + \tfrac{\alpha}{2}\, y(1 - y).$$

αBB for SIP with nonconvex lower level



αBB for SIP with nonconvex lower level

Properties of $\tilde{g}(x,y;\alpha) \;=\; g(x,y) + \frac{\alpha}{2}y(1-y)$:

$$\forall\ \alpha \in \mathbb{R}: \quad \tilde{g}(x,0;\alpha) = g(x,0), \quad \tilde{g}(x,1;\alpha) = g(x,1),$$

$$\forall\ \alpha \geq 0,\ \ y \in [0,1]: \quad \tilde{g}(x,y;\alpha) \;\geq\; g(x,y),$$

$$\forall\ \alpha \geq \max_{(x,y)\in X\times Y} D_y^2 g(x,y),\ \ x \in X: \quad \tilde{g}(x,\cdot\,;\alpha) \text{ concave on } Y.$$

αBB for SIP with nonconvex lower level

So choose $$\alpha \;\geq\; \max\left(0, \max_{(x,y)\in X\times Y} D_y^2 g(x,y)\right).$$

Catch:

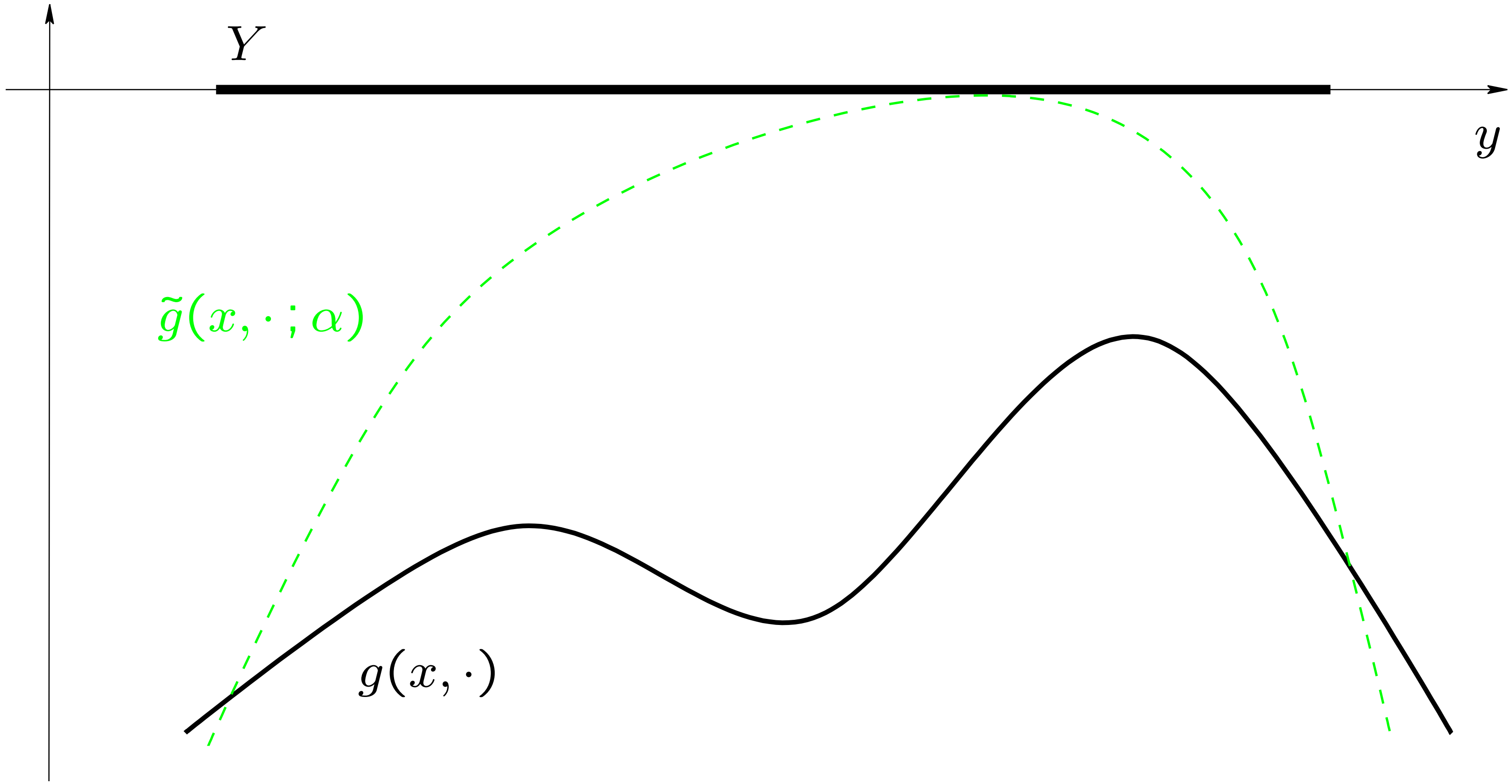
We have to solve *another* global optimization problem for α !

Advantage:

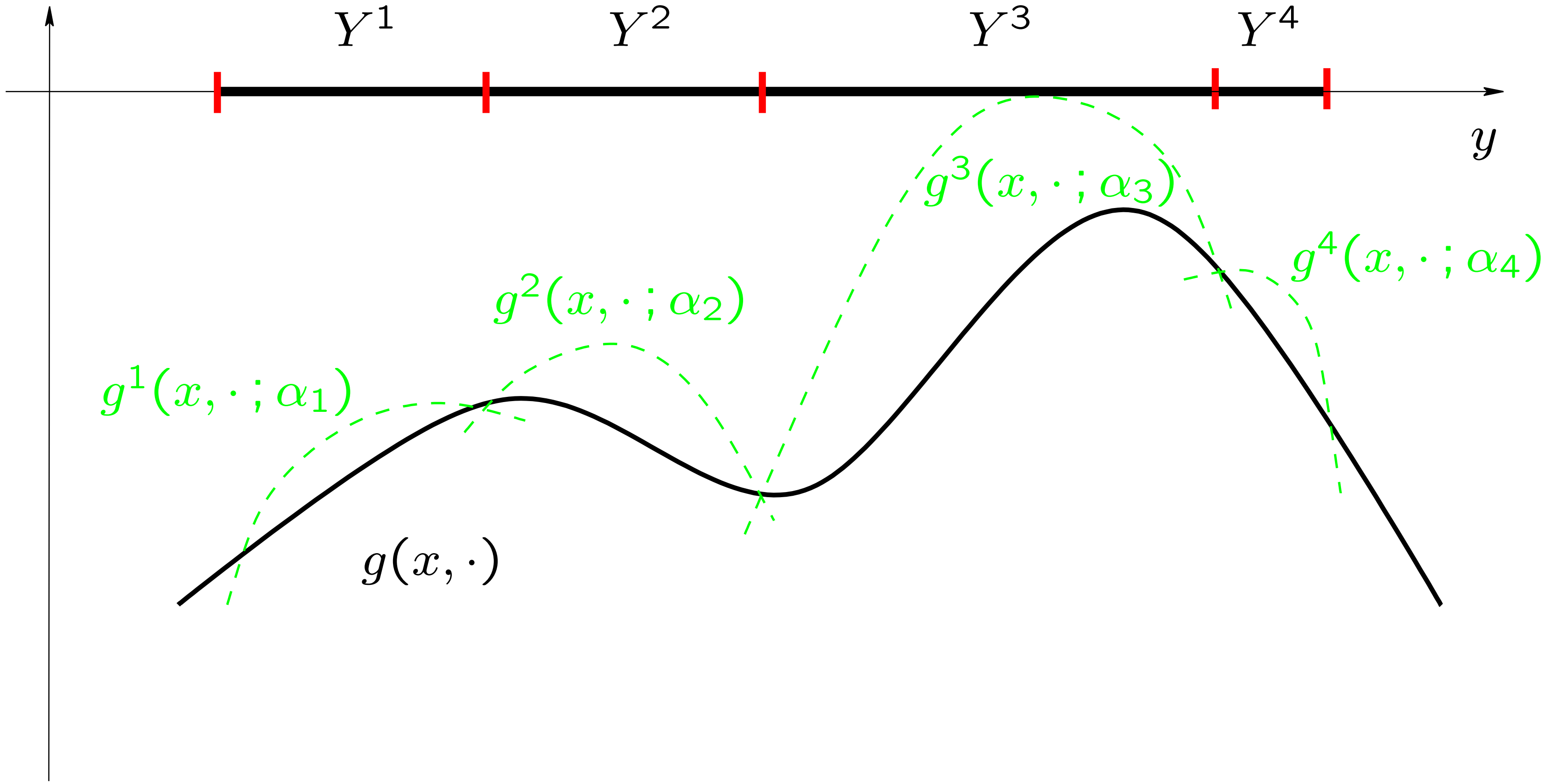
We only need *some* upper bound for its optimal value,

and this can be generated with, for example, the methods of interval arithmetic (Neumaier 1990, Hansen 1992, Floudas 2000, Matlab toolbox INTLAB by Siegfried Rump).

αBB for SIP with nonconvex lower level



Adaptive subdivision of Y



Adaptive subdivision of Y

For $N \in \mathbb{N}$ let $0 = \eta^0 < \eta^1 < ... < \eta^{N-1} < \eta^N = 1$ define a partition of $Y = [0, 1]$, that is, with $K = \{1, ..., N\}$ and

$$Y^k \;=\; [\eta^{k-1}, \eta^k], \qquad k \in K,$$

let

$$Y \;=\; \bigcup_{k \in K} Y^k\,.$$

Trivial, but very useful fact:

$$g(x, y) \le 0 \quad \text{for all } y \in Y$$

is equivalent to

$$g(x, y) \le 0 \quad \text{for all } y \in Y^k,\ k \in K.$$

Adaptive subdivision of Y

For given subdivision points $E = \left\{\eta^k | \ k \in K\right\}$, on each Y^k the adaptive convexification algorithm constructs a concave overestimator

$$g^k(x,y) \ = \ g(x,y) + \frac{\alpha_k}{2}(y - \eta^{k-1})(\eta^k - y),$$

by αBB,

by MPCC reformulation solves

$$SIP_{\alpha BB}(E,\alpha): \quad \min_{x \in X} f(x) \quad \text{s.t.} \quad x \in M_{\alpha BB}(E,\alpha),$$

with

$$M_{\alpha BB}(E,\alpha) \ = \ \{ x \in \mathbb{R}^n | \ g^k(x,y) \leq 0 \text{ for all } y \in Y^k, \ k \in K \},$$

and then adaptively refines the subdivision of Y.

Adaptive subdivision of Y

Lemma (Floudas/St. 2006):

For all $E = \left\{\eta^k |\ k \in K\right\}$ and $\alpha_k\,,\ k \in K$, with

$$\alpha_k \ \geq \ \max\left(0, \max_{(x,y)\in X\times Y^k} D_y^2 g(x,y)\right) \qquad (\star)$$

we have $M_{\alpha BB}(E,\alpha) \subset M$.

Hence any solution concept for $SIP_{\alpha BB}(E,\alpha)$ produces feasible points for SIP!

Adaptive subdivision of Y

If the inequality in $(\star)$ is strict, then each lower level problem $Q^k(x)$ has a *unique* solution $y^k(x)$.

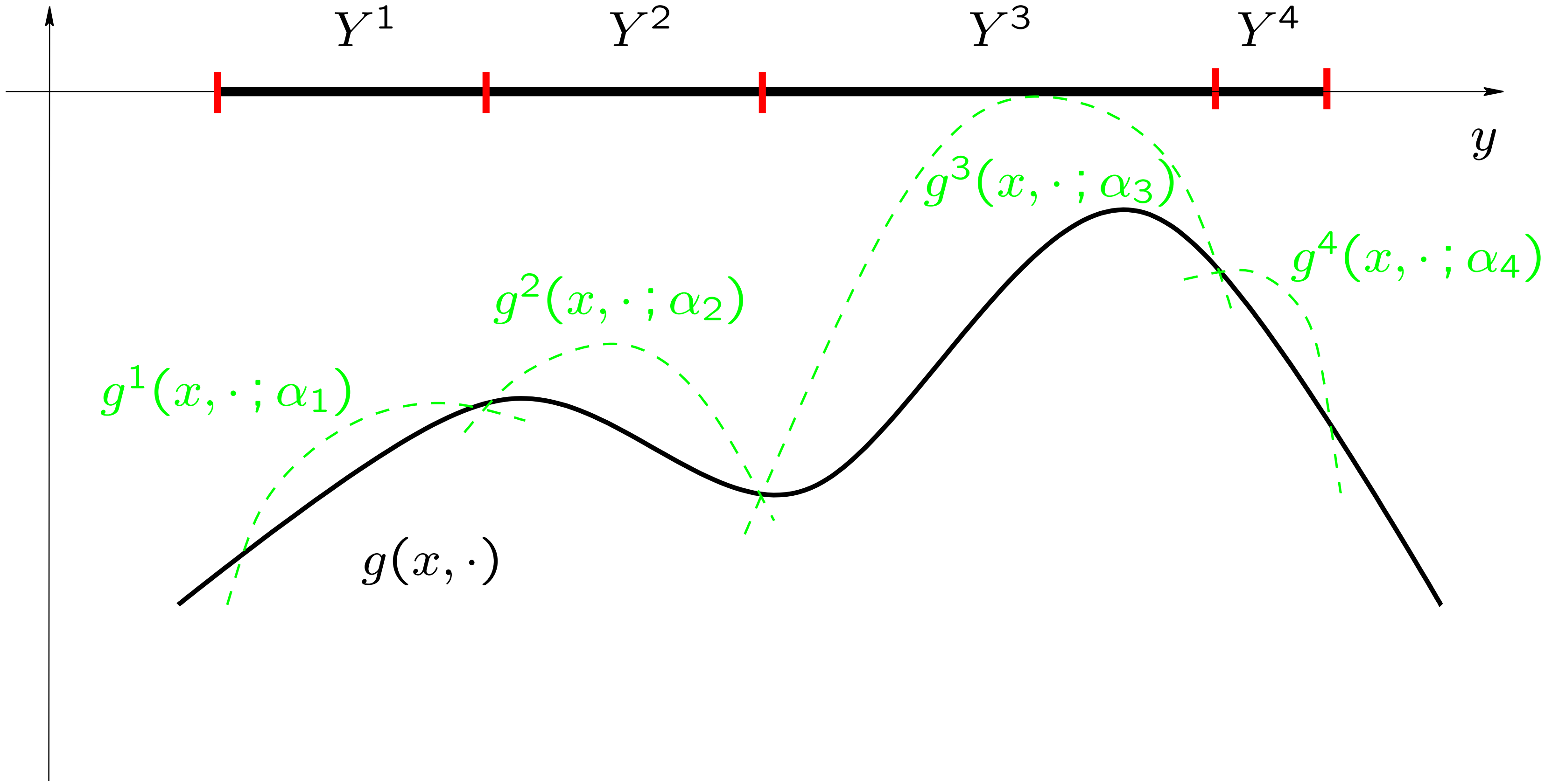
Hence $\bar{x} \in M_{\alpha BB}(E, \alpha)$ possesses finite active index sets

$$K_0(\bar{x}) \;=\; \{\, k \in K |\; g^k(\bar{x}, y^k(\bar{x})) = 0 \,\},$$

$$Y_0^{\alpha BB}(\bar{x}) \;=\; \{\, y^k(\bar{x}) |\; k \in K_0(\bar{x}) \,\}.$$

Adaptive refinement of the partition:
If solution $\bar{x}$ of $SIP_{\alpha BB}(E, \alpha)$ is not $\varepsilon-$stationary for SIP, solve refined problem $SIP_{\alpha BB}(E \cup Y_0^{\alpha BB}(\bar{x}), \tilde{\alpha})$.

Adaptive subdivision of Y



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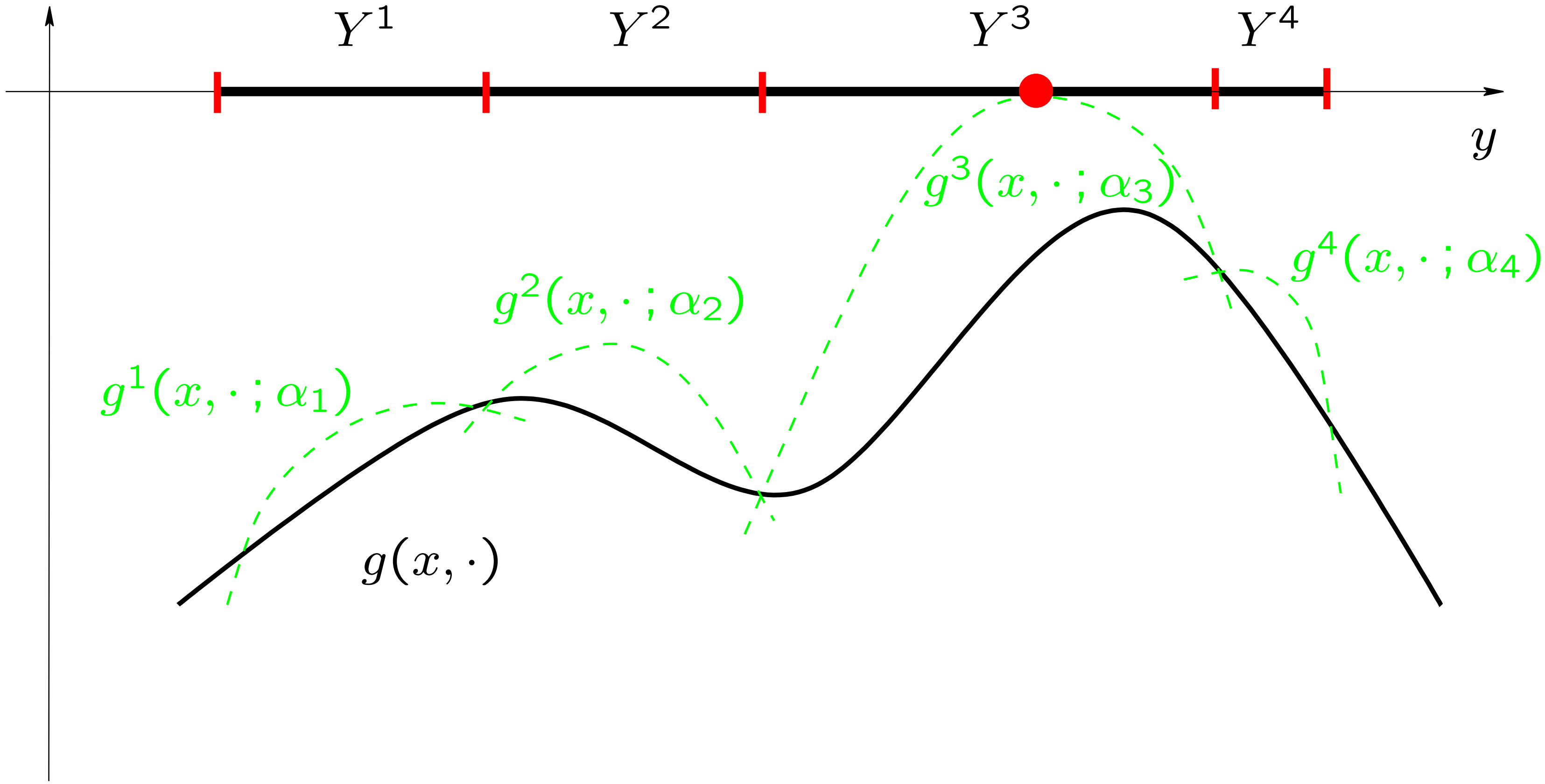

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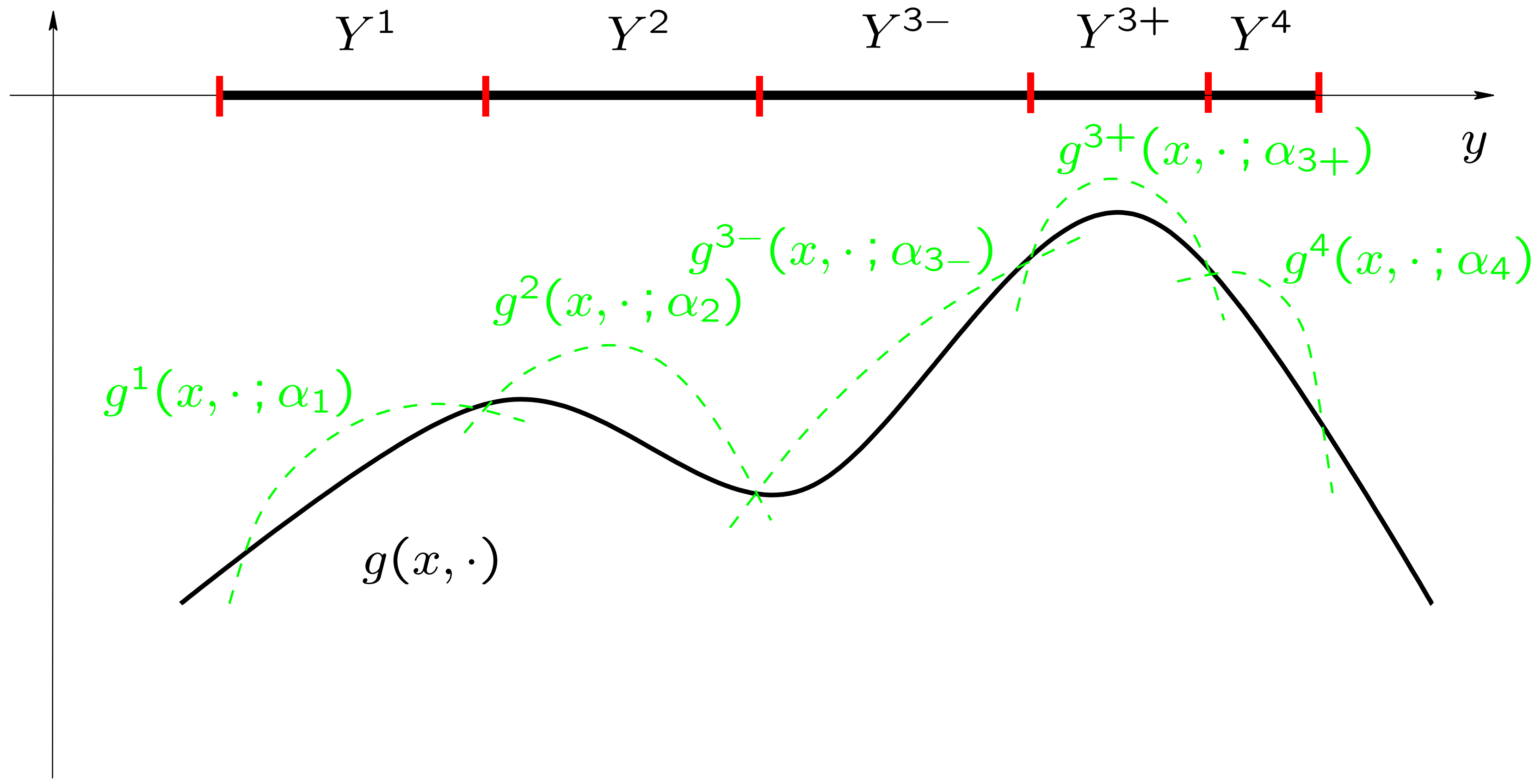


Convergence

Theorem (Floudas/St. 2006):
The adaptive convexification algorithm is well defined and finite.

Proposition (Floudas/St. 2006):
Let $E_N = \{k/N|\ k = 1, ..., N\}$. Then $\bar{x} \in M$ with $\varphi(\bar{x}) < 0$ is a feasible point of $SIP_{\alpha BB}(E_N, \alpha)$ for all

$$N \ \geq \ \frac{1}{2}\sqrt{\frac{\max_{k \in K} \alpha_k}{|\varphi(\bar{x})|}} \ .$$

Convergence

Proposition (Floudas/St. 2006):
Let $(E^\nu)_\nu$ be a sequence of subdivisions of $Y = [0,1]$ with $E^\nu \subset E^{\nu+1}$, let the sets $M^\nu := M_{\alpha BB}(E^\nu, \alpha^\nu)$ be defined by the described refinement steps, and let v^ν denote the optimal value of $SIP_{\alpha BB}(E^\nu, \alpha^\nu)$, $\nu \in \mathbb{N}$. Then we have:

(i) $M^1 \subset M^2 \subset \ \dots \ \subset M^\nu \subset M^{\nu+1} \subset \ \dots \ \subset M$.

(ii) If SIP is solvable, then the v^ν converge to an upper bound for the optimal value of SIP.

(iii) Each sequence $(x^\nu)_\nu$ with $x^\nu \in X \cap M^\nu$, $\nu \in \mathbb{N}$, is feasible for SIP and possesses an accumulation point $x^\star$. Each such accumulation point is also feasible for SIP, and $f(x^\star)$ is and upper bound for the optimal value of SIP.

Numerical results: Chebyshev Approximation

Approximation of $\sin(\pi y)$ by quadratic function on [0,1],

$$\min_{x \in \mathbb{R}^3} \| \sin(\pi y) - (x_3 y^2 + x_2 y + x_1) \|_{\infty,[0,1]},$$

is, with the error function

$$e(x, y) \; = \; \sin(\pi y) - (x_3 y^2 + x_2 y + x_1),$$

equivalent to

$$\min_{x \in \mathbb{R}^4} x_4 \quad \text{s.t.} \quad \pm e(x, y) \leq x_4 \quad \text{for all } y \in [0, 1].$$

We look for a solution in $X = [x^\ell, x^u]$
with $x^\ell = (-1, 3, -5, -1)^\top$ and $x^u = (1, 5, -3, 3)^\top$.

Numerical results: Chebyshev Approximation

Convexification parameters $\alpha_\pm$ for

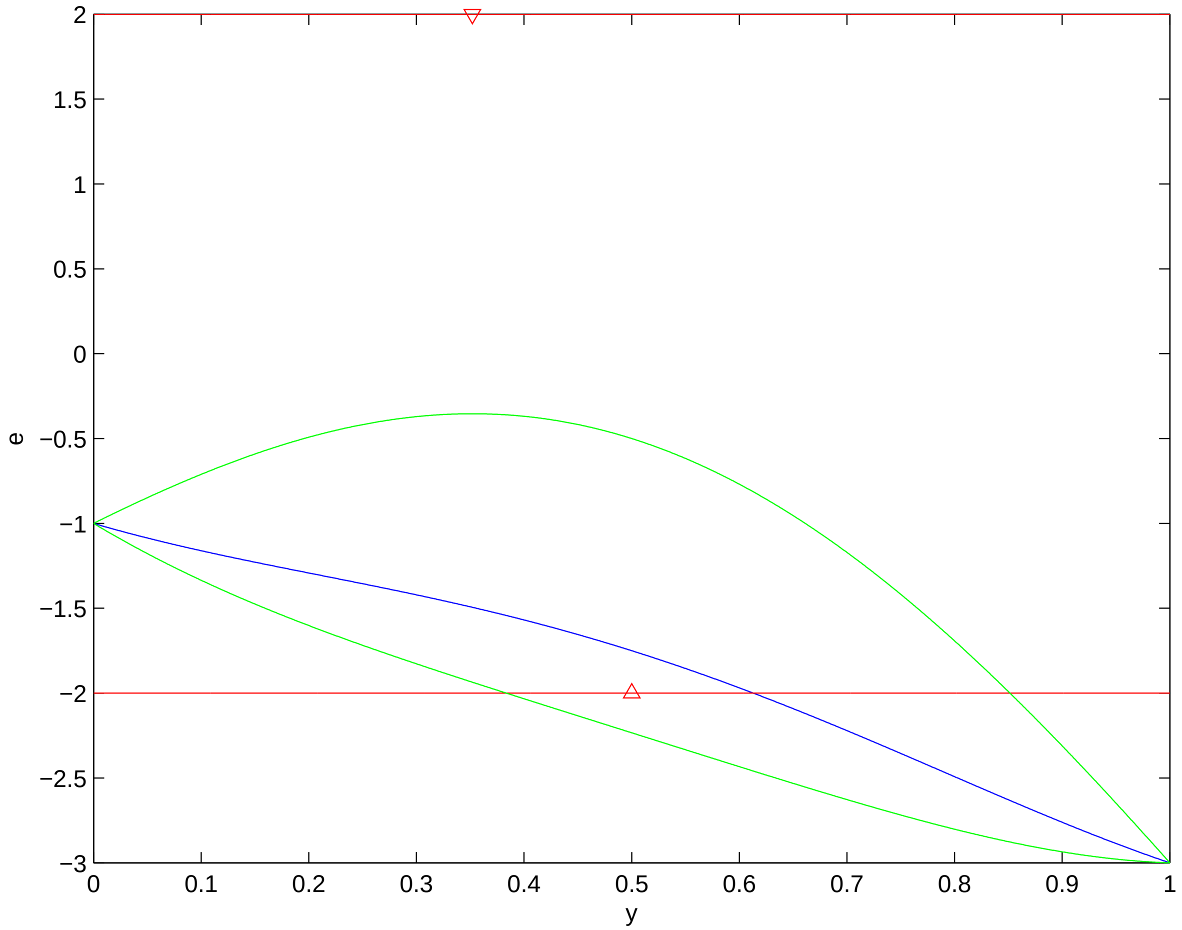
$$g_\pm(x,y) \;=\; \pm e(x,y) - x_4$$

on $X \times Y$ are, due to

$$D_y^2 g_\pm(x,y) = \mp\left(\pi^2 \sin(\pi y) + 2x_3\right)\,,$$

$\alpha_+ = 10$ and $\alpha_- = \pi^2 - 6$.

Numerical results: Chebyshev Approximation



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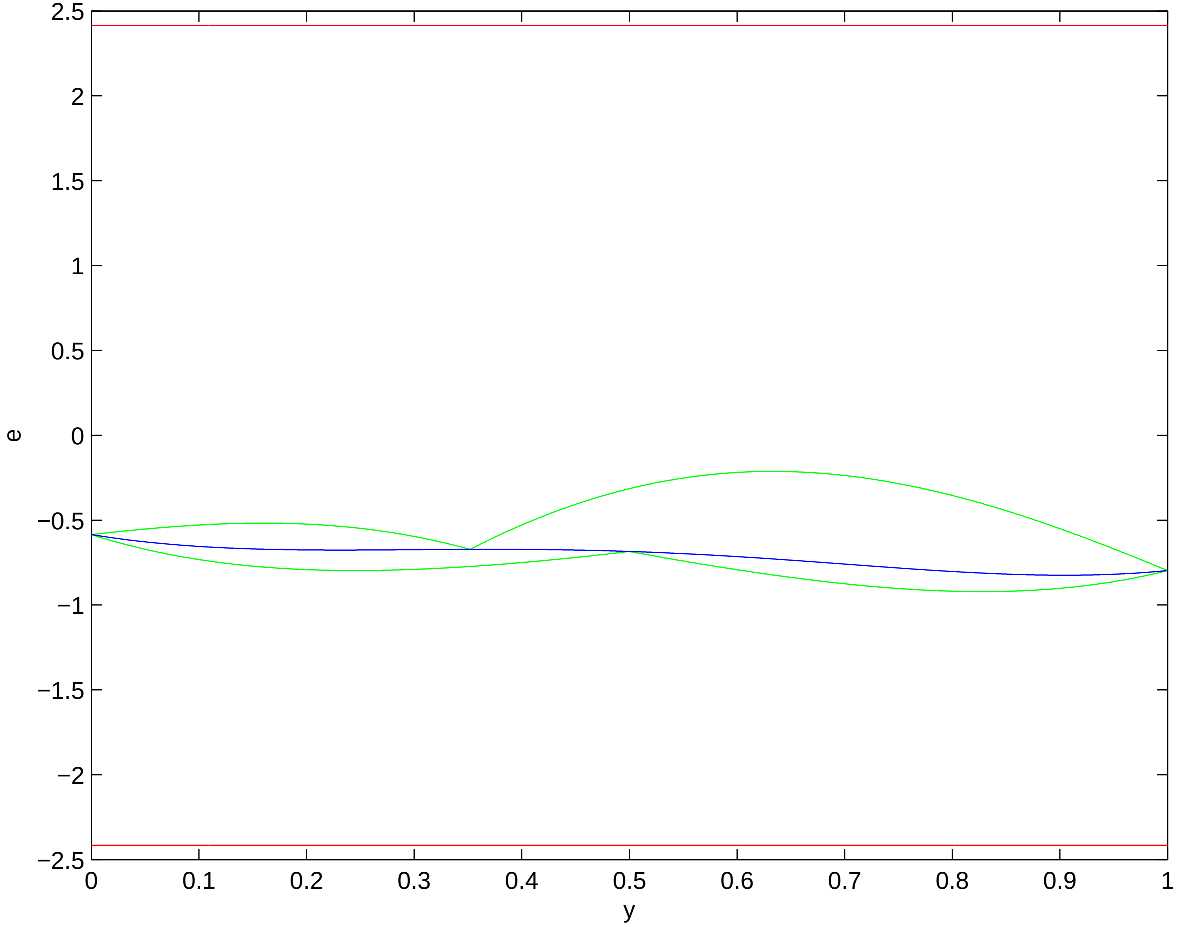

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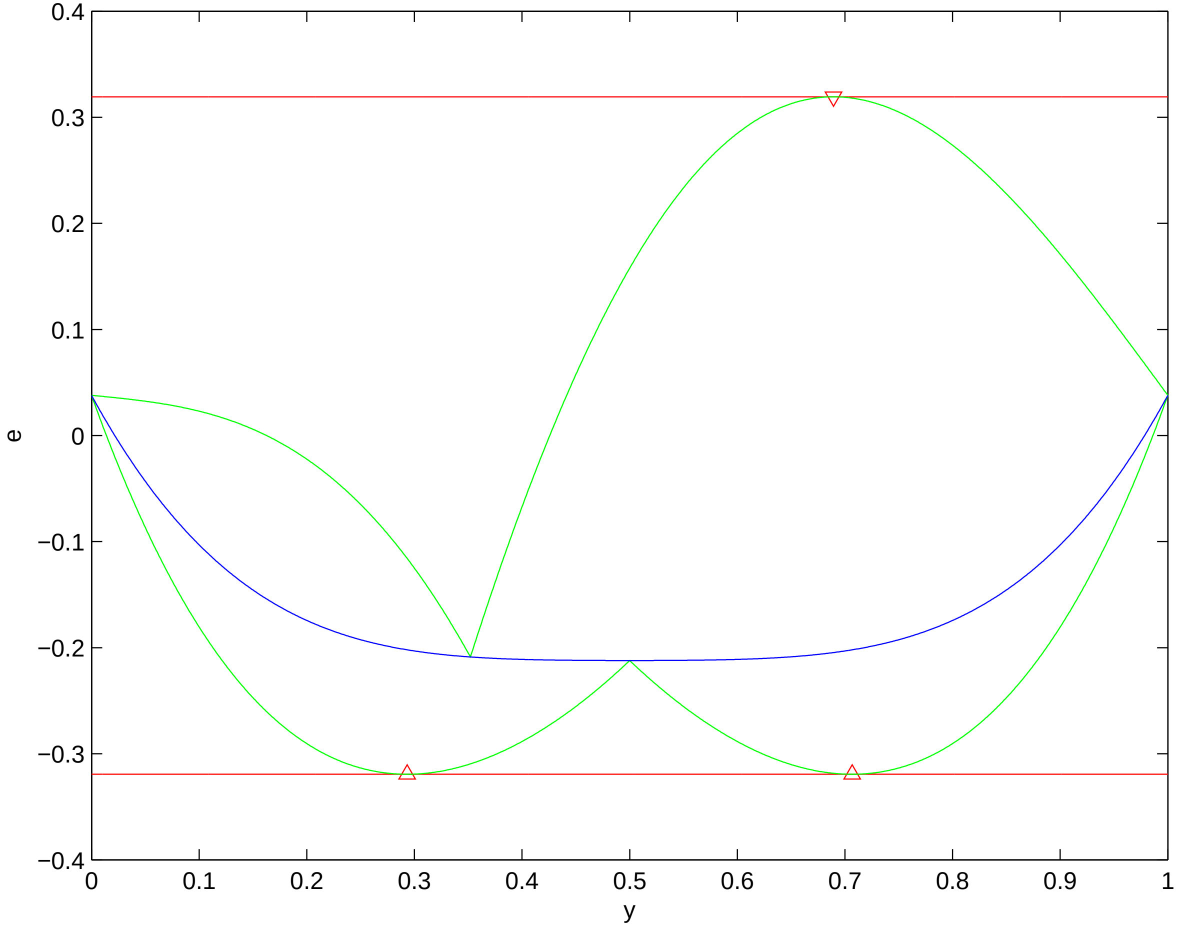

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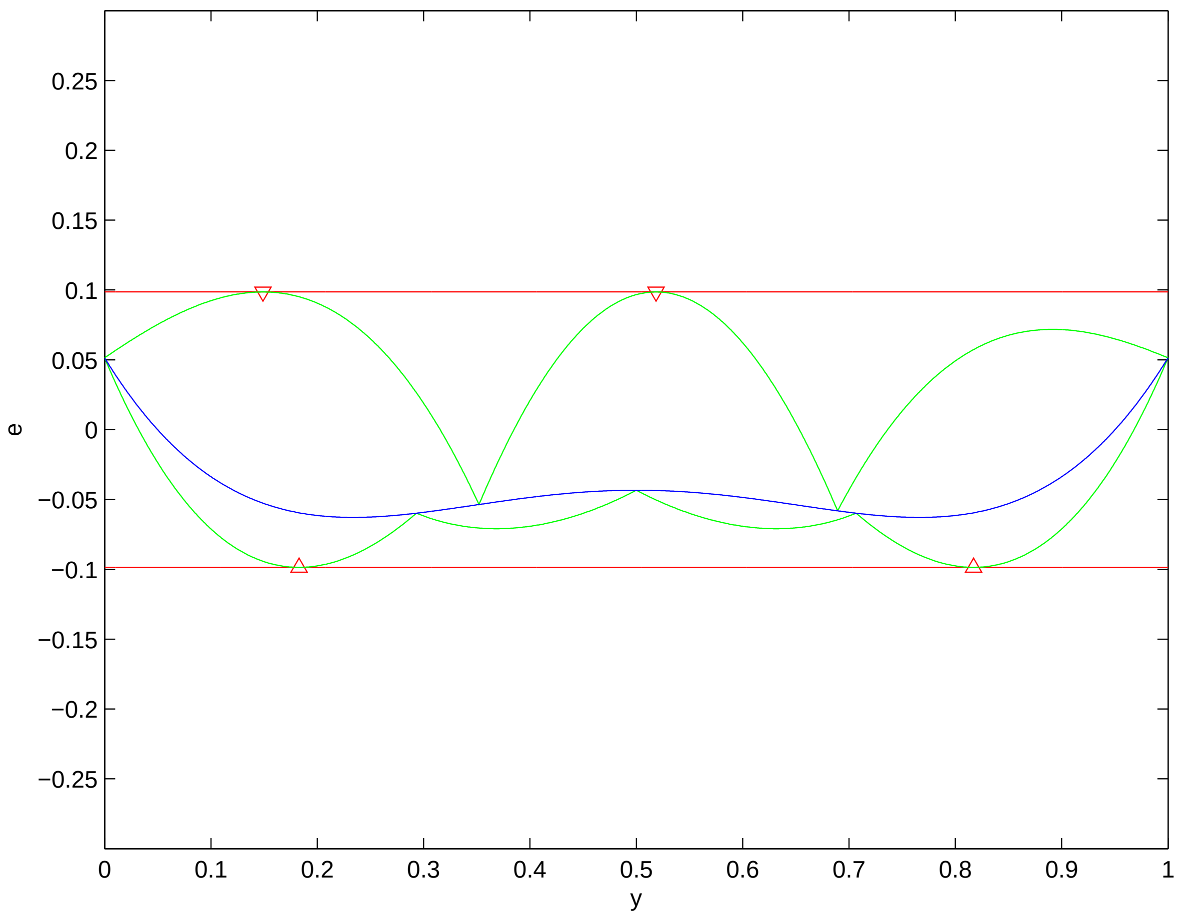
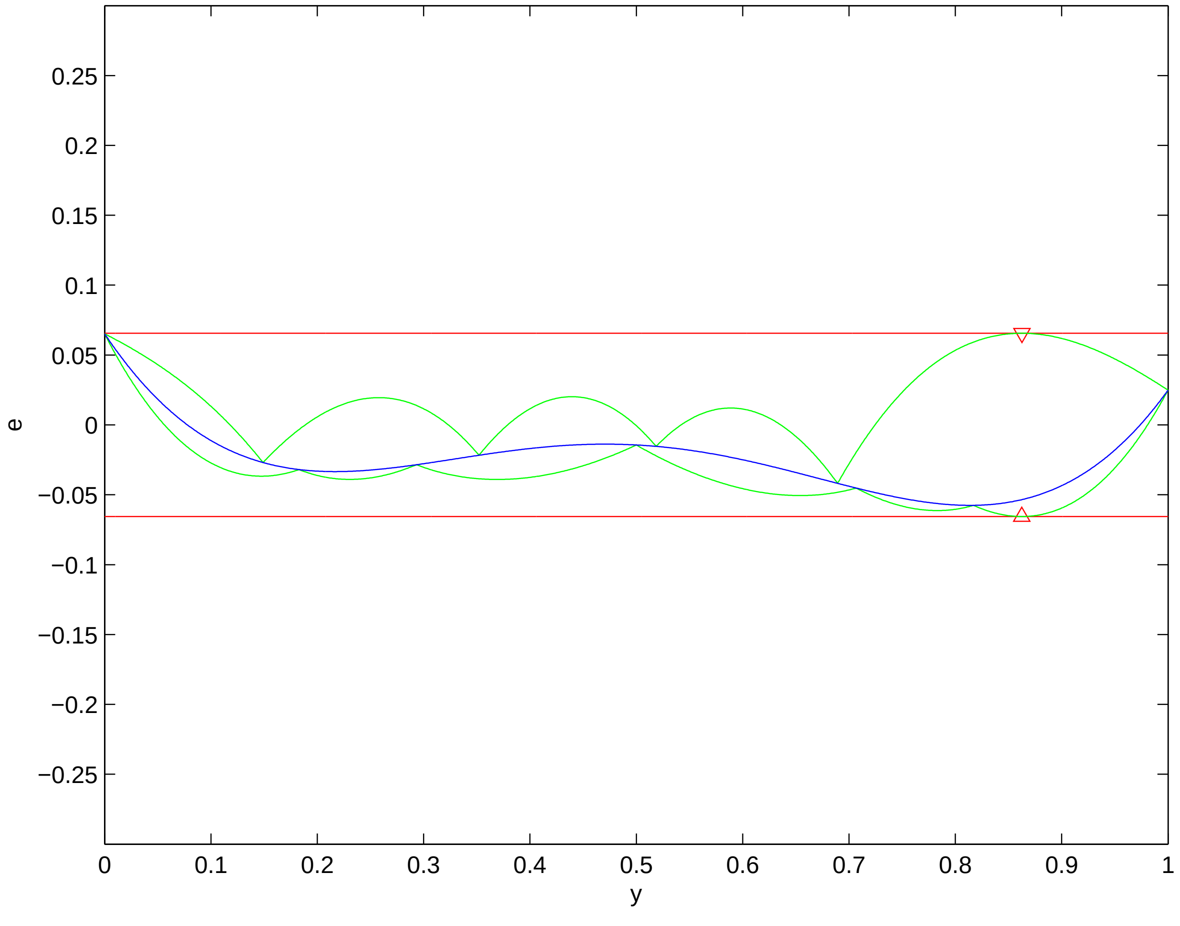
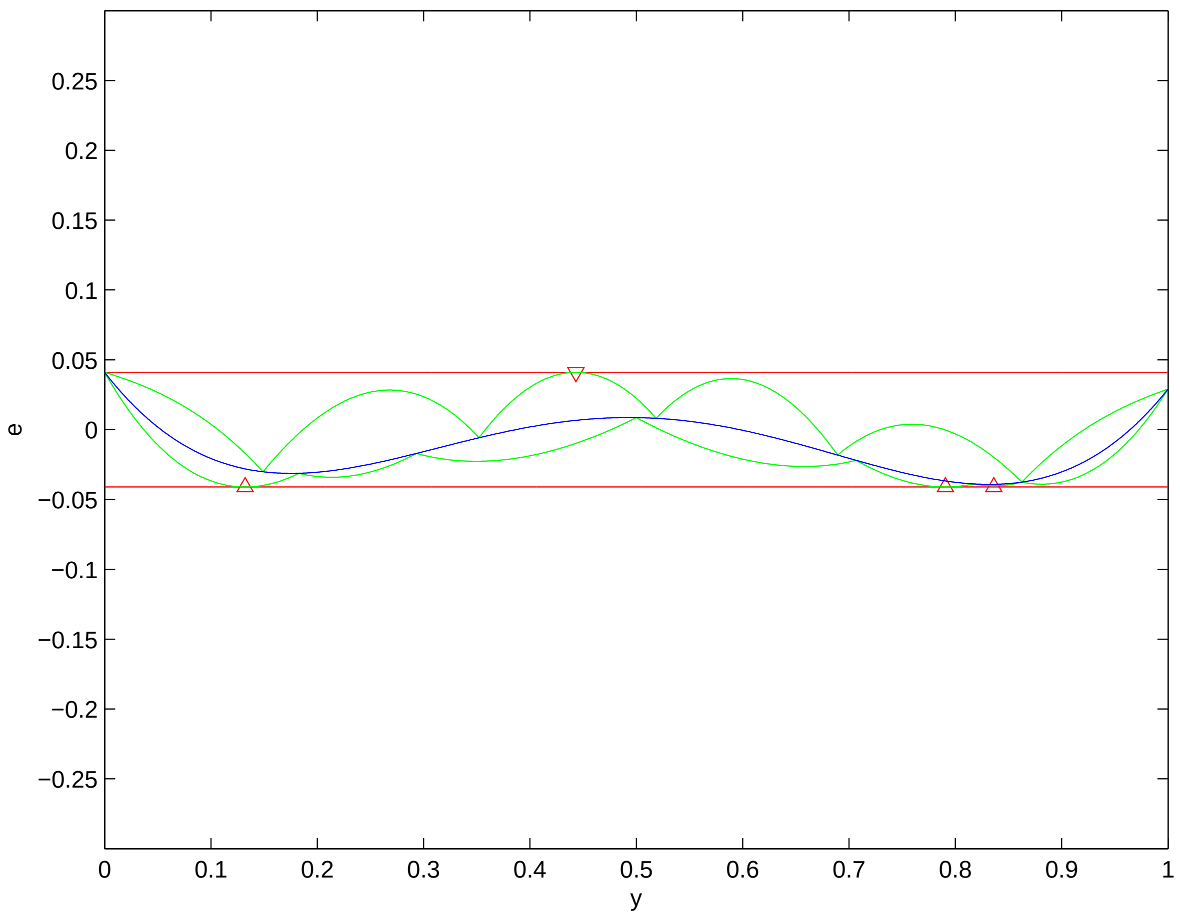


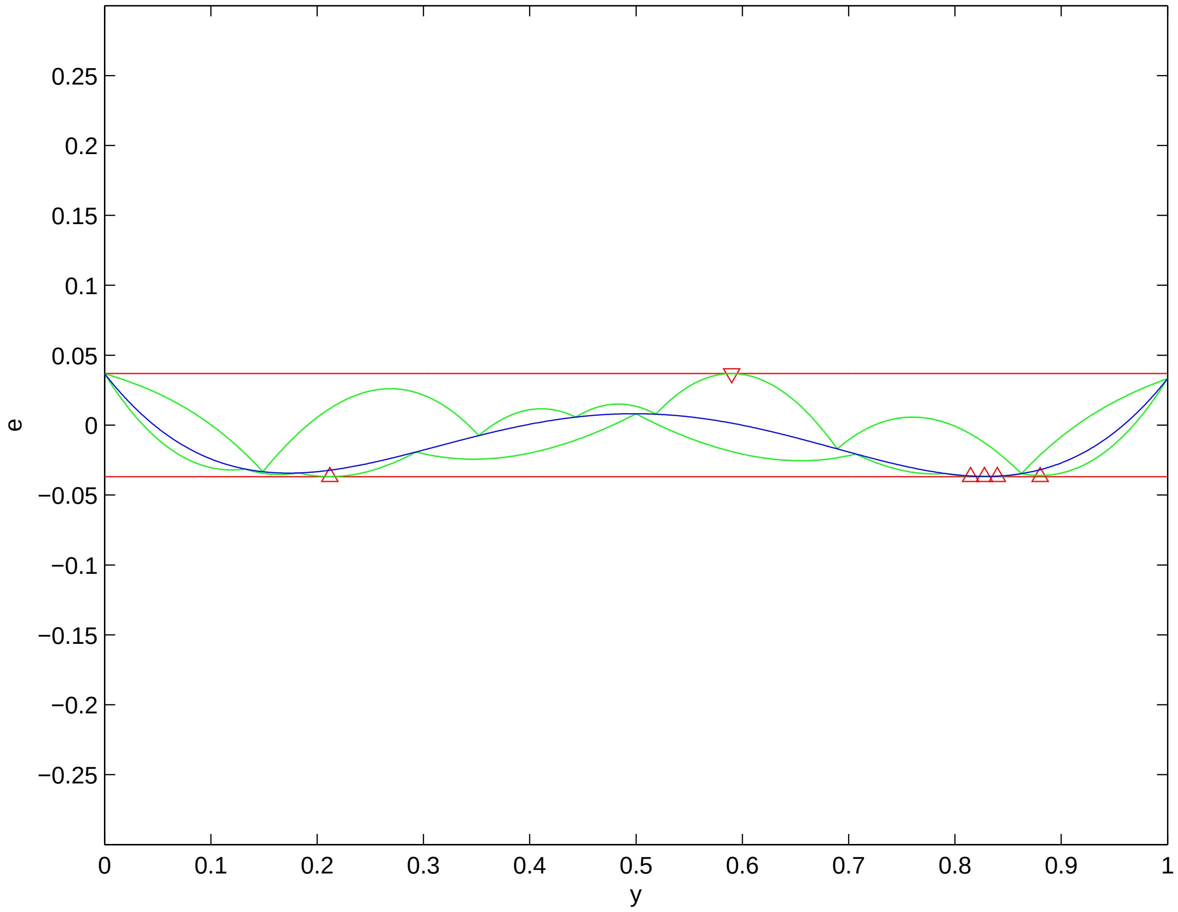
Numerical results: Chebyshev Approximation



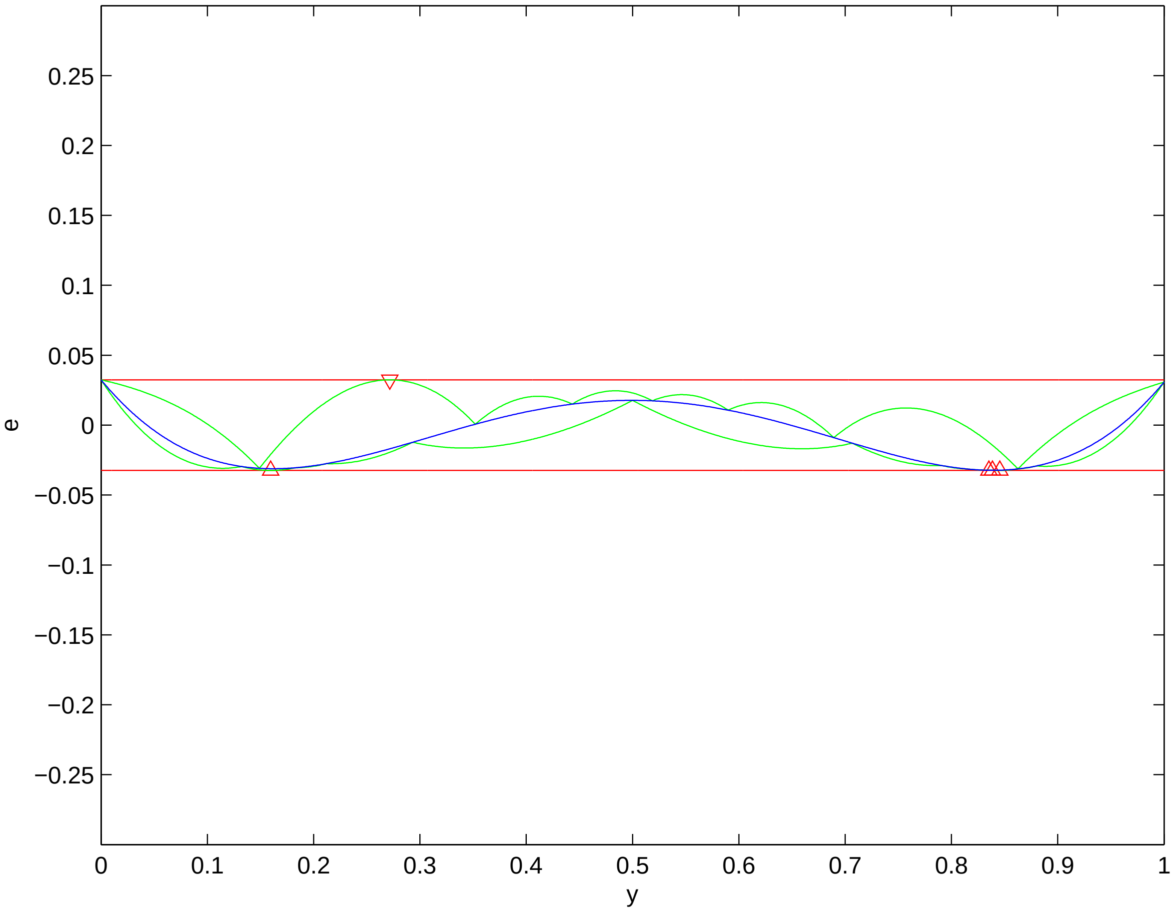
Numerical results: Chebyshev Approximation



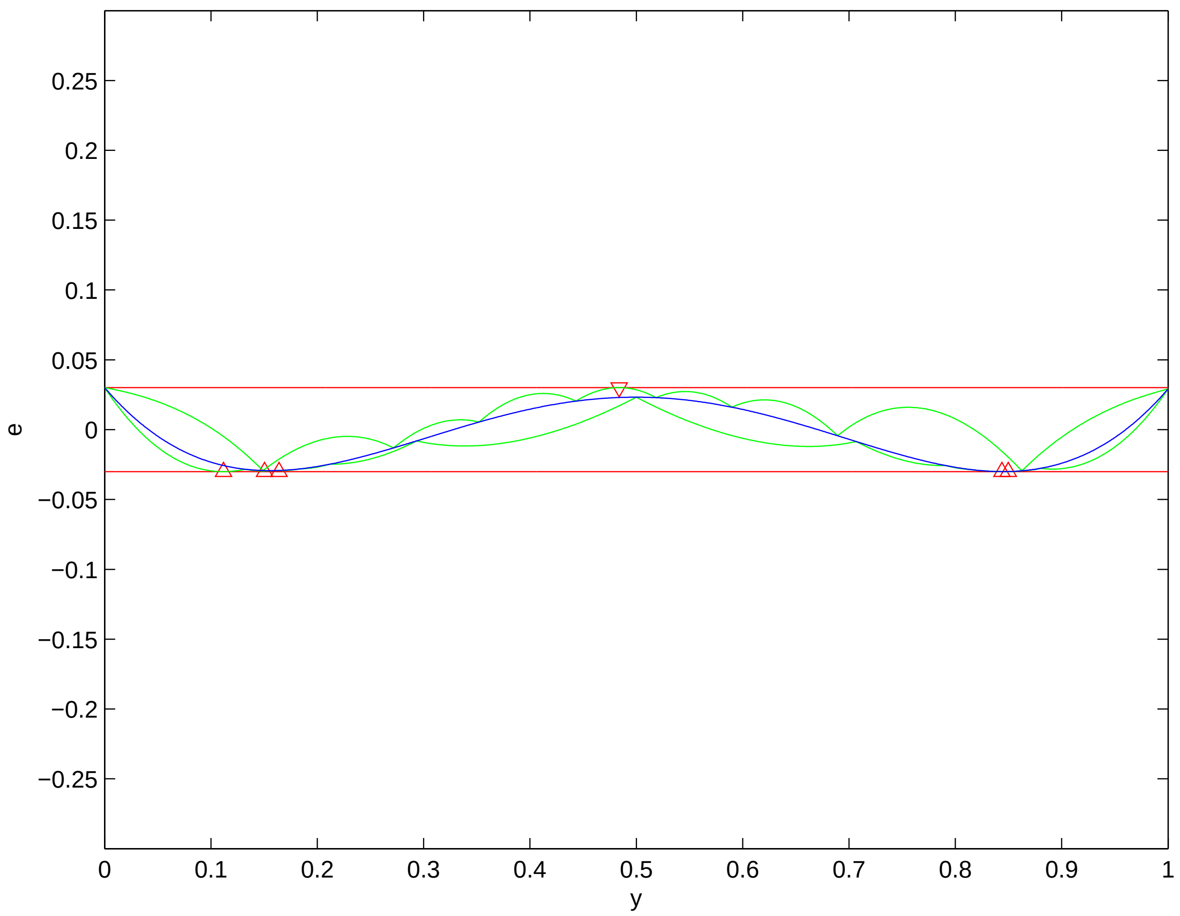
Numerical results: Chebyshev Approximation



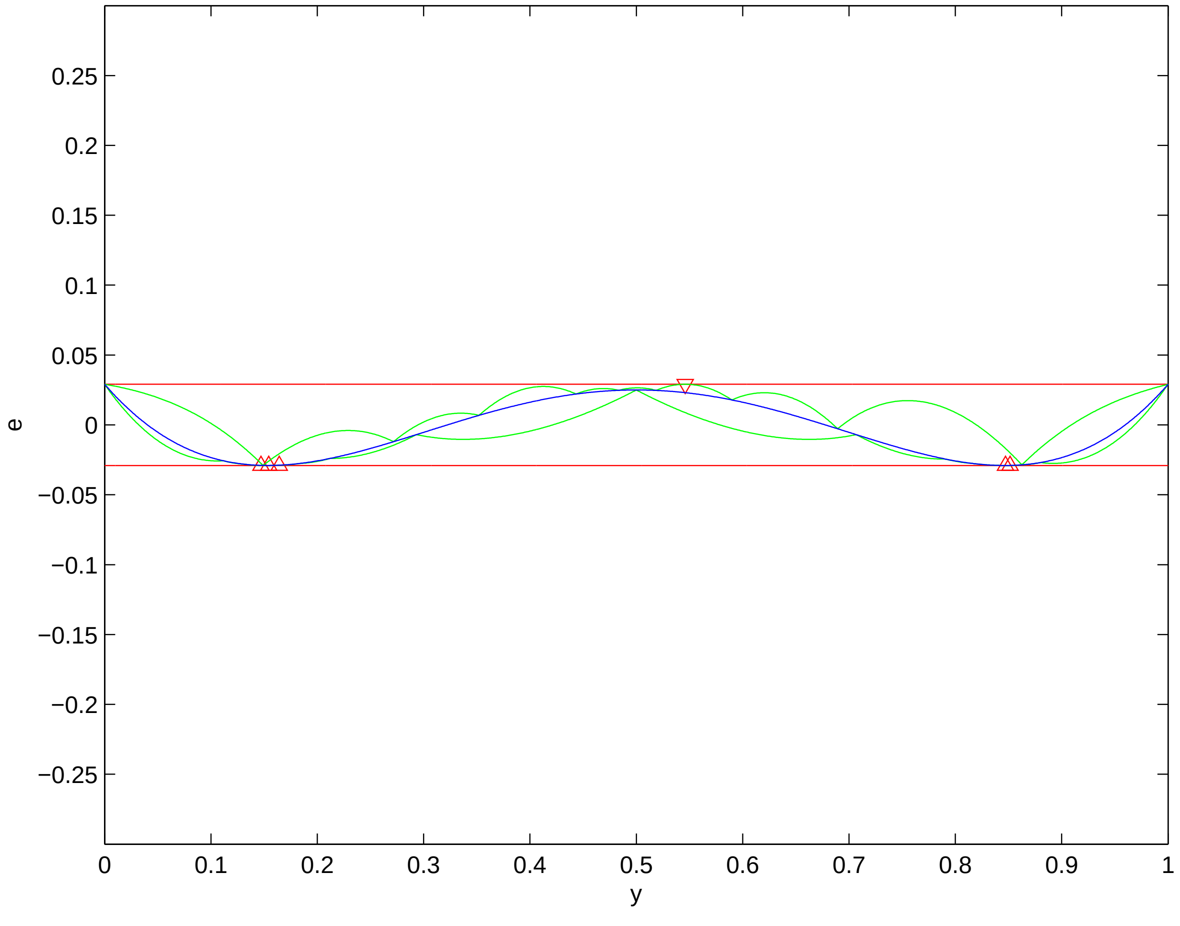
Numerical results: Chebyshev Approximation



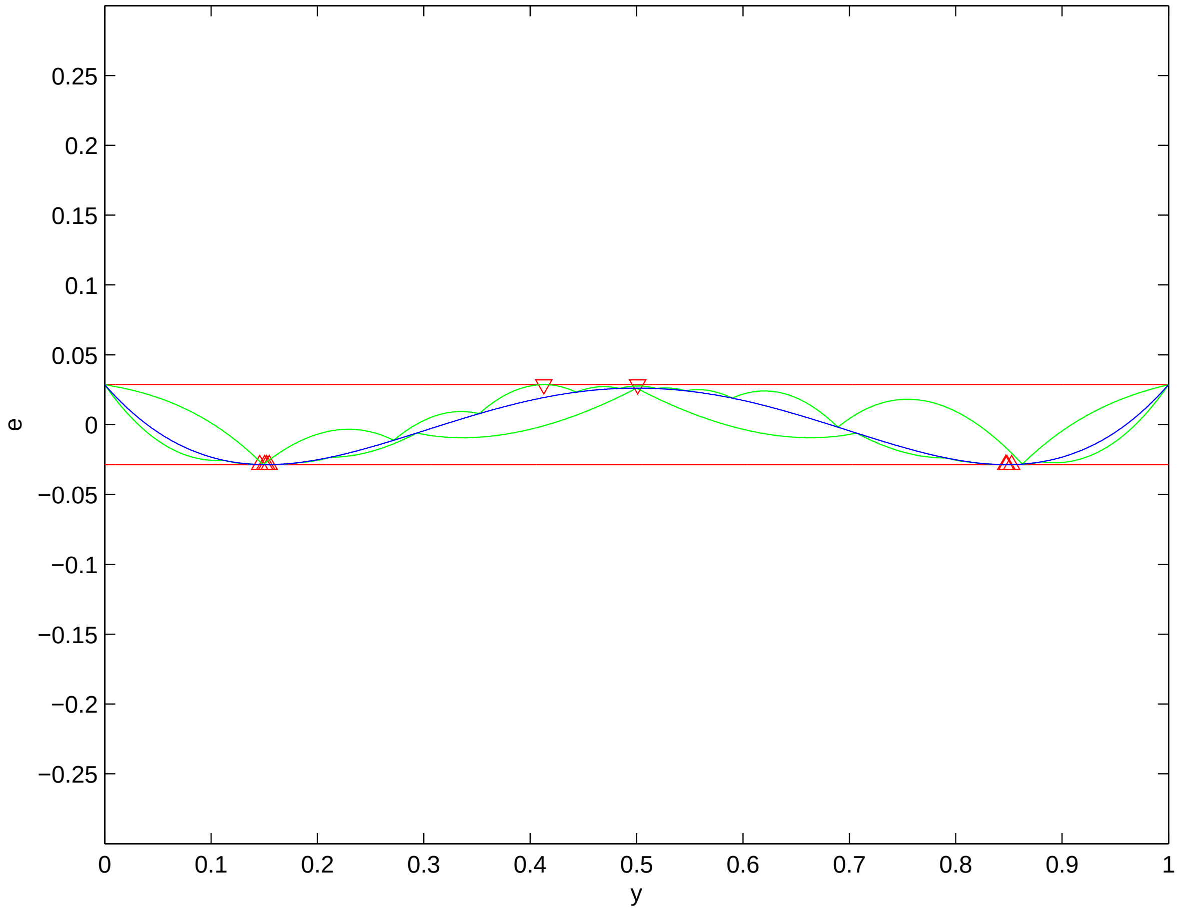
Numerical results: Chebyshev Approximation



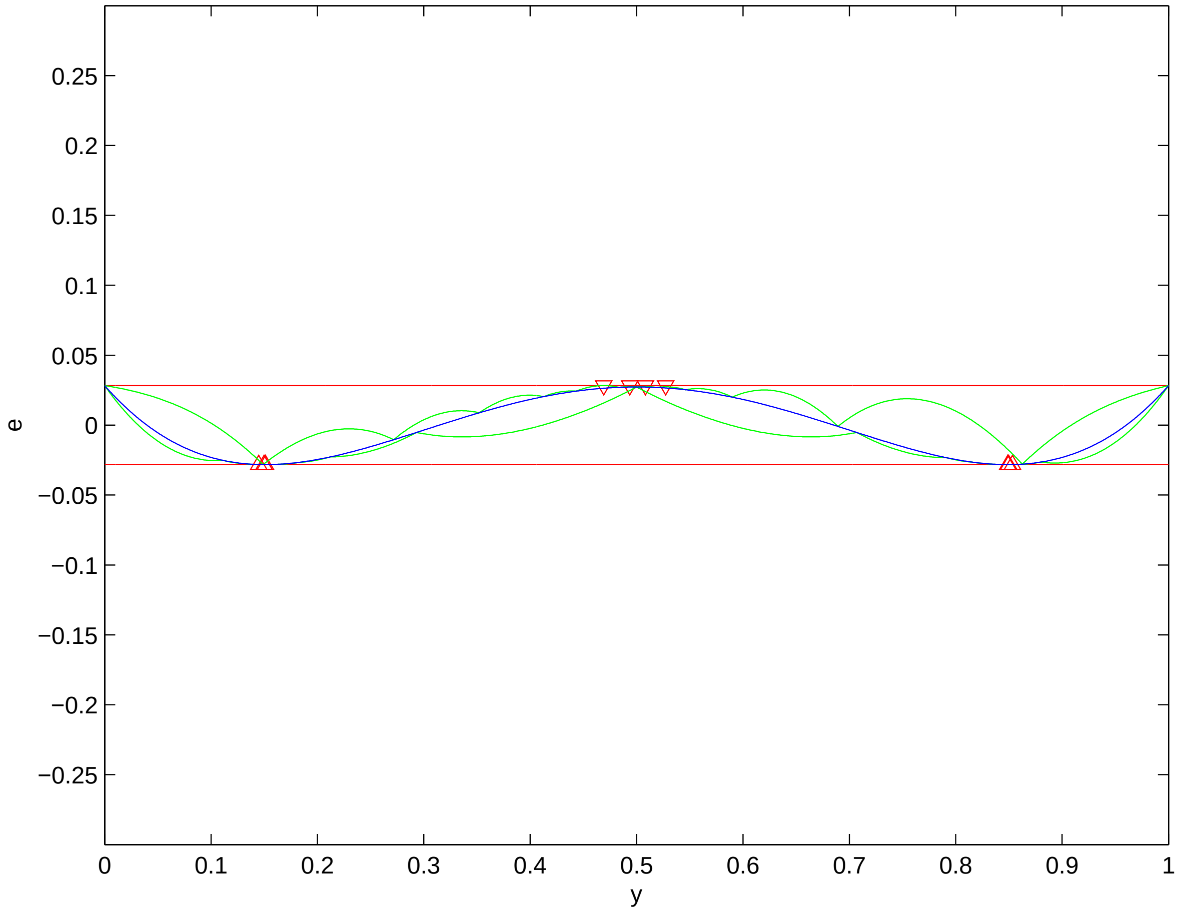
Numerical results: Chebyshev Approximation



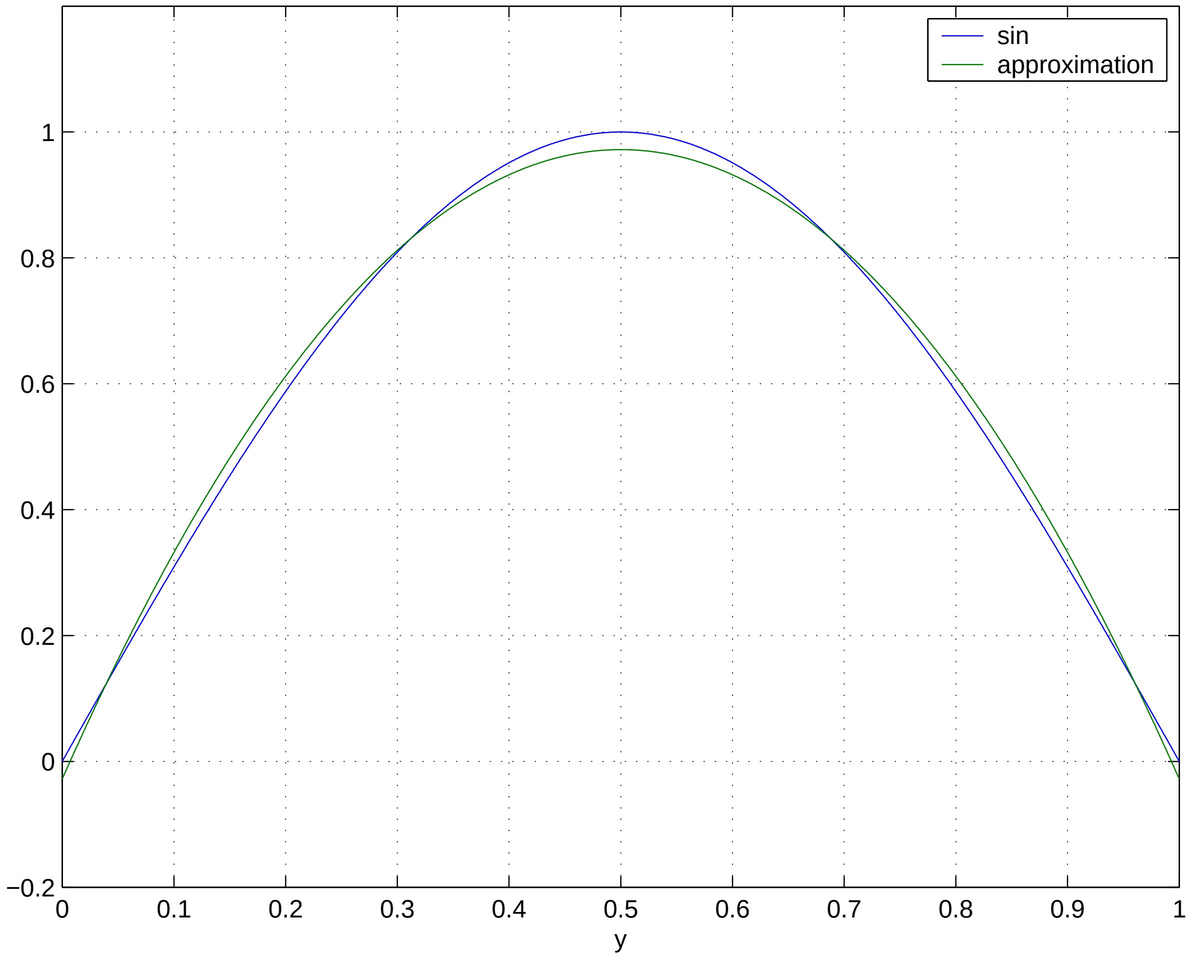
Numerical results: Chebyshev Approximation



Numerical results: Chebyshev Approximation



Numerical results: Chebyshev Approximation



Numerical results: Chebyshev Approximation

Effort to compute a 10^{-3}–stationary point:
11 iterations, 17.044 CPU seconds.

Upper bound for optimal error: 0.028265

With the last set of subdivision points E the discretized problem

$$\min_{x \in X} f(x) \quad \text{s.t.} \quad g(x,\eta) \leq 0, \ \eta \in E,$$

is *linear* in x and can thus be solved to global optimality.

Its optimal value 0.028005 is a lower bound for the optimal error, and we obtain a *certificate* for computation of the optimal error within a precision of 10^{-3}!

Numerical results: Chebyshev Approximation

Convexification parameters $\alpha_\pm$ on subintervals $[\eta^\ell, \eta^u] \subset [0,1]$ yield *tighter* concave overestimators:

$$\begin{aligned}
\alpha_+(\eta^\ell, \eta^u) &= \mathsf{max}\Big(0,\ -\pi^2 \,\mathsf{min}(\,\mathsf{sin}(\pi\eta^\ell),\ \mathsf{sin}(\pi\eta^u)\,) + 10\Big) \\
\alpha_-(\eta^\ell, \eta^u) &= \mathsf{max}\Big(0,\ \pi^2\theta(\eta^\ell, \eta^u) - 6\Big)
\end{aligned}$$

with

$$\theta(\eta^\ell, \eta^u) = \begin{cases} \mathsf{sin}(\pi\eta^u), & \eta^u \le 0.5 \\ 1, & \eta^\ell < 0.5 < \eta^u \\ \mathsf{sin}(\pi\eta^\ell), & 0.5 \le \eta^\ell\,. \end{cases}$$

Numerical results: Chebyshev Approximation

Effort to compute a 10^{-3}–stationary point:
7 iterations, 5.689 CPU seconds.

Upper bound for optimal error: 0.028011

Lower bound for optimal error: 0.028005

Certificate for globally optimal error within precision of 10^{-5}.

Numerical results: Design Centering

Consider the problem to inscribe the boundary of a disk,

$$B(x) \;=\; \{\, z(x,y) |\; y \in [0, 2\pi]\,\}$$

with

$$z(x,y) \;=\; \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + x_3 \begin{pmatrix} \cos(y) \\ \sin(y) \end{pmatrix} \right),$$

with maximal radius x_3 into the container

$$C \;=\; \{\, z \in \mathbb{R}^2 |\; c_i(z) \le 0,\; i \in I\,\},$$

that is, the SIP

$$\max_{x \in \mathbb{R}^3} x_3 \quad \text{s.t.} \quad g_i(x,y) := c_i(z(x,y)) \le 0 \text{ for all } y \in Y := [0, 2\pi],\; i \in I.$$

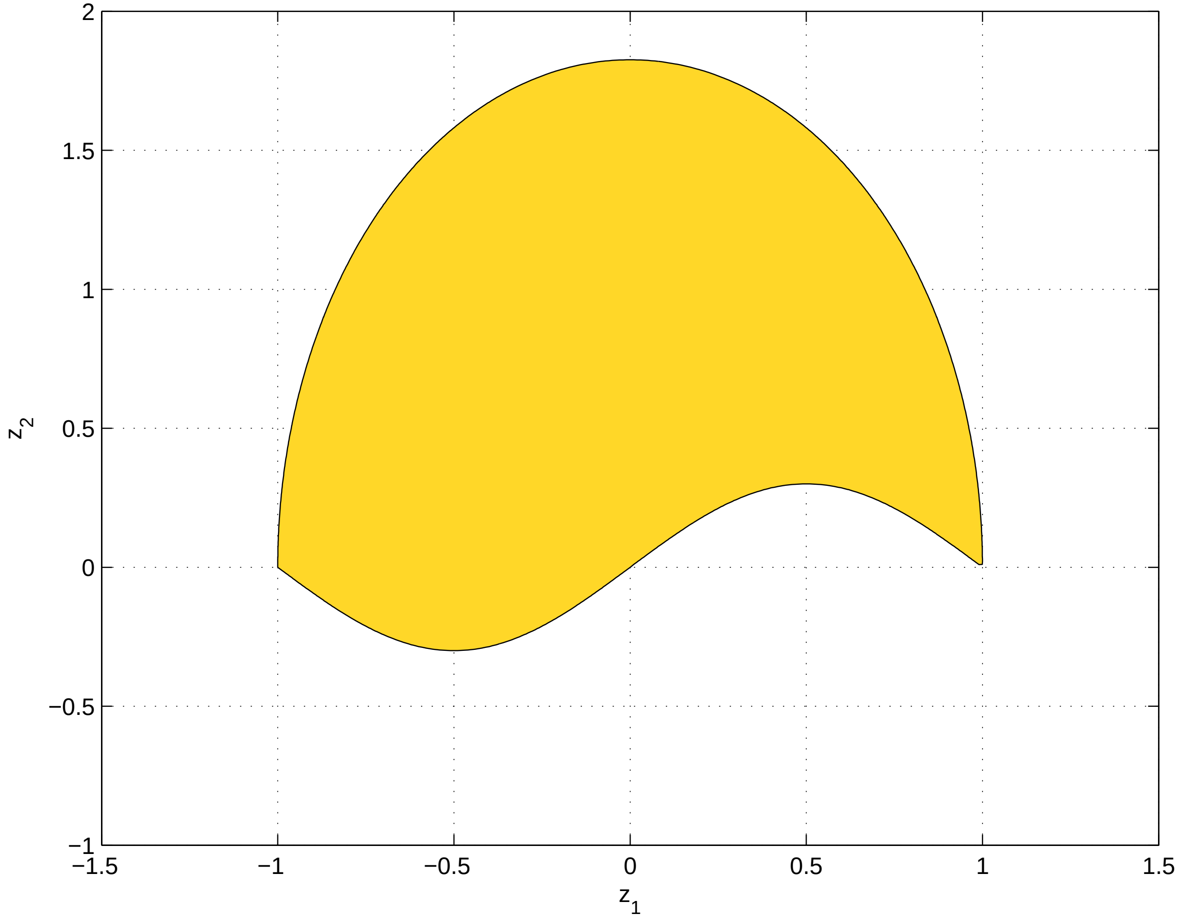
Numerical results: Design Centering

Convexification parameters α_i come, for example, from the coarse estimate

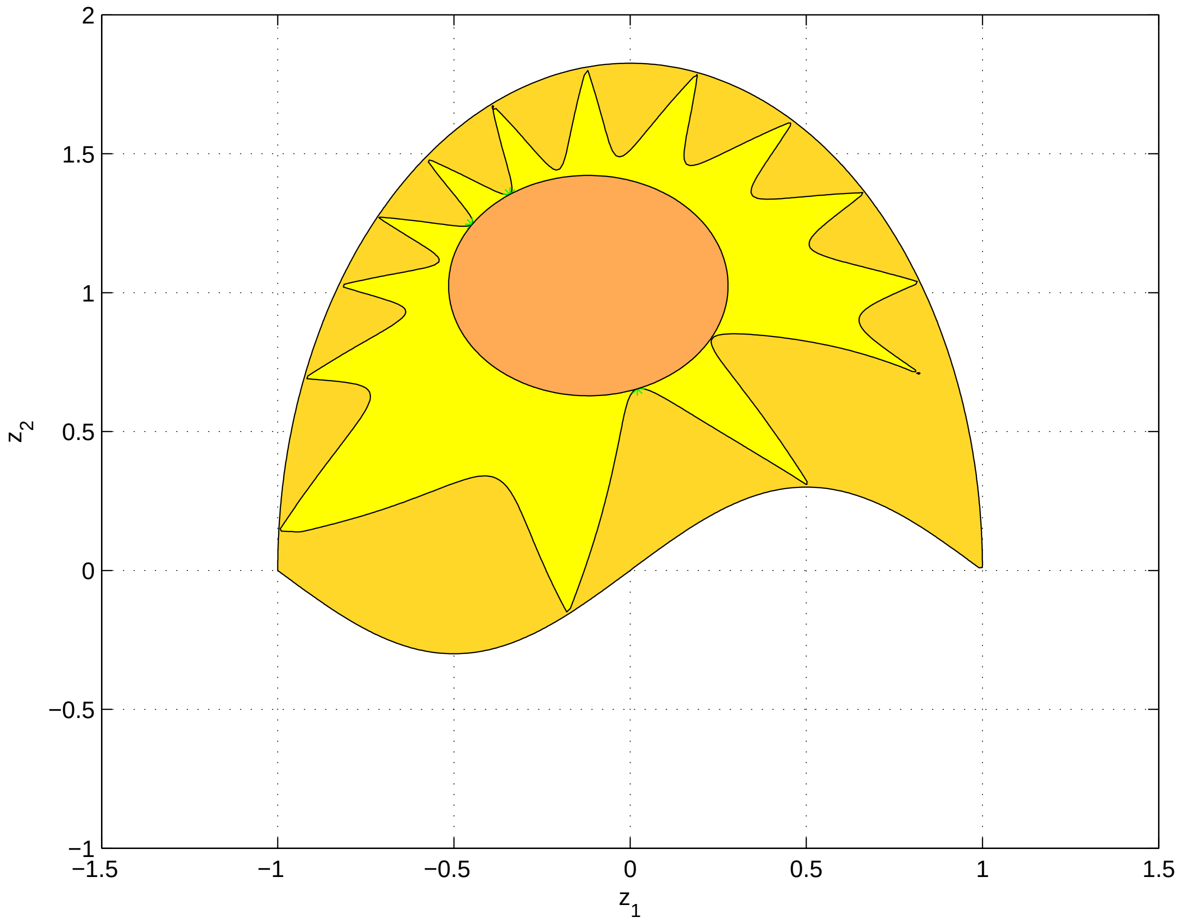
$$\max_{(x,y)\in X\times Y} D^2 g_i(x,y)$$

$$\leq \max_{(x,y)\in X\times Y} \left(x_3^2\, \lambda_{max} \left(D^2 c_i(z(x,y)) \right) + x_3 ||\nabla c_i(z(x,y))||_2 \right).$$

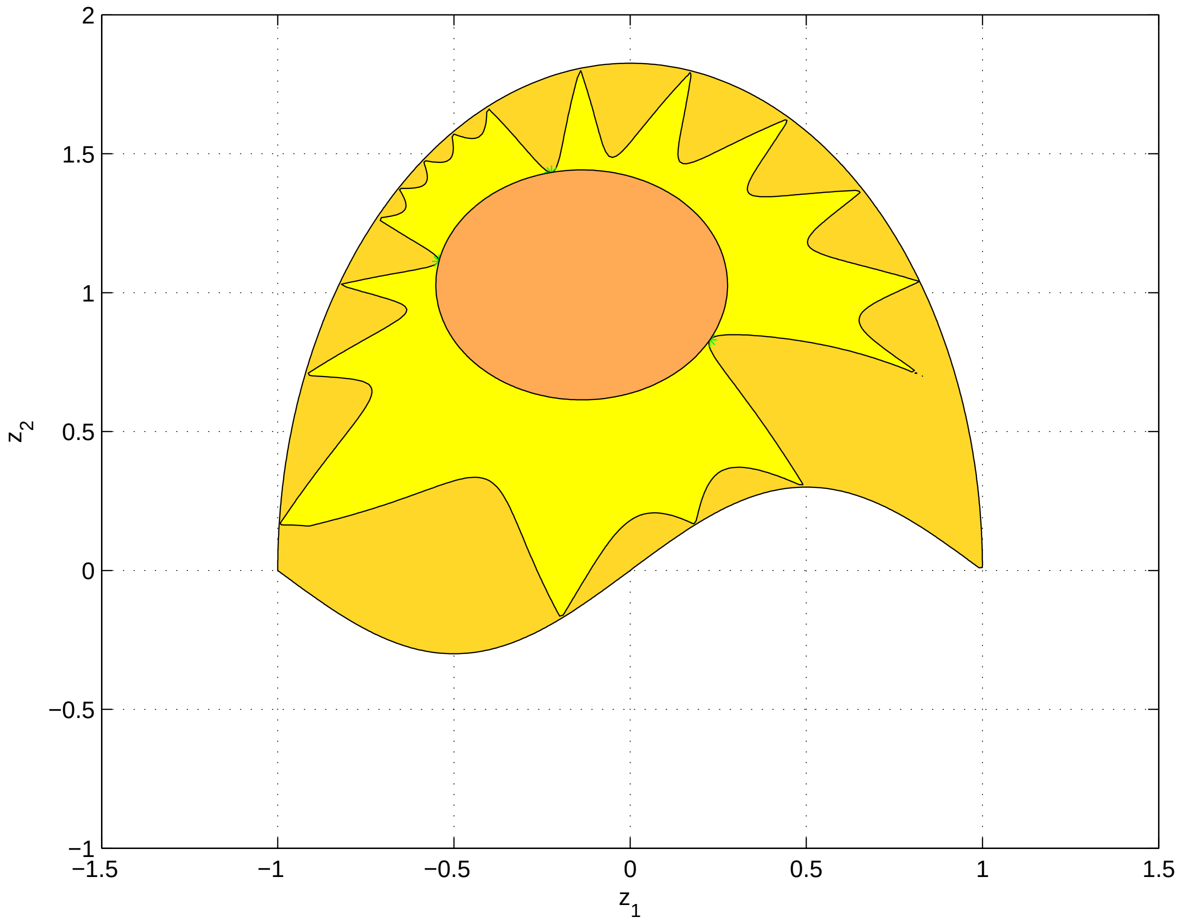
Numerical results: Design Centering



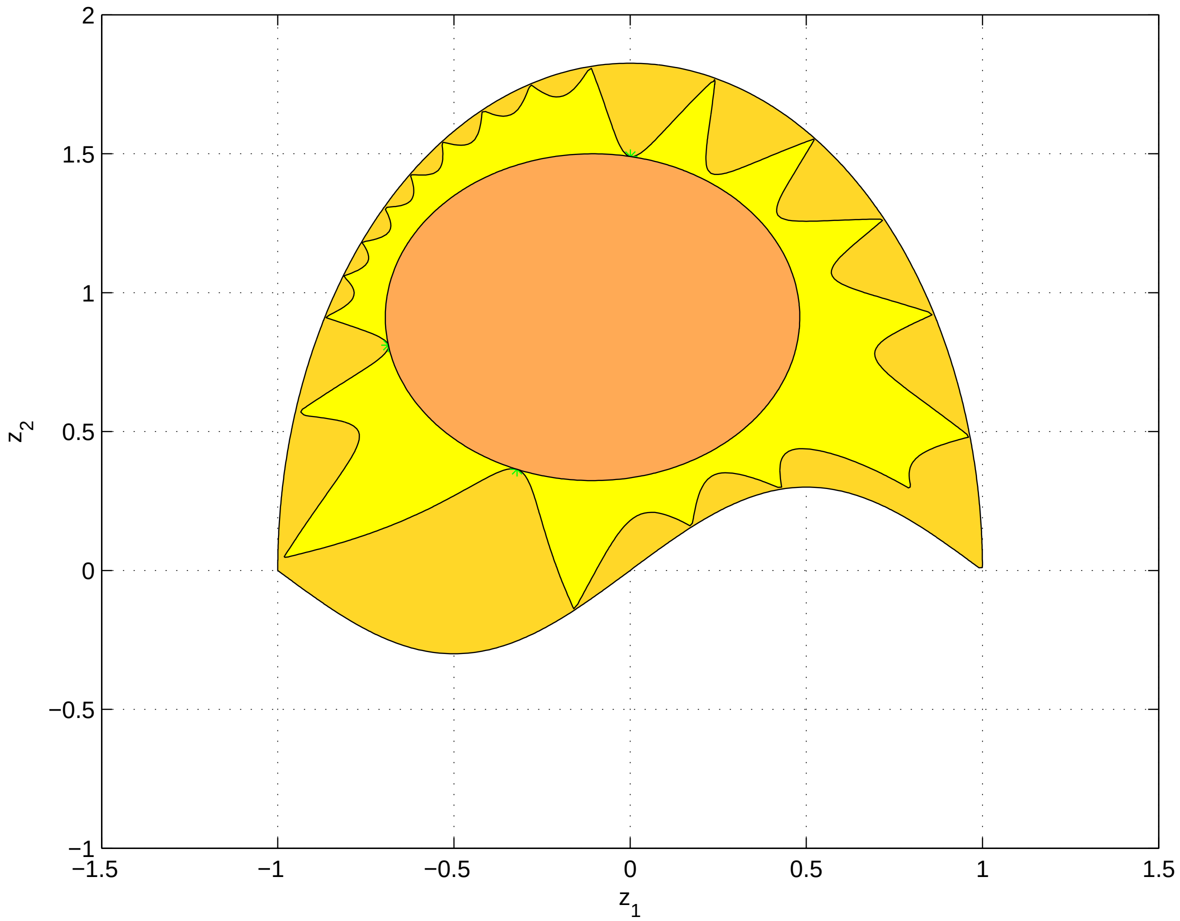
Numerical results: Design Centering



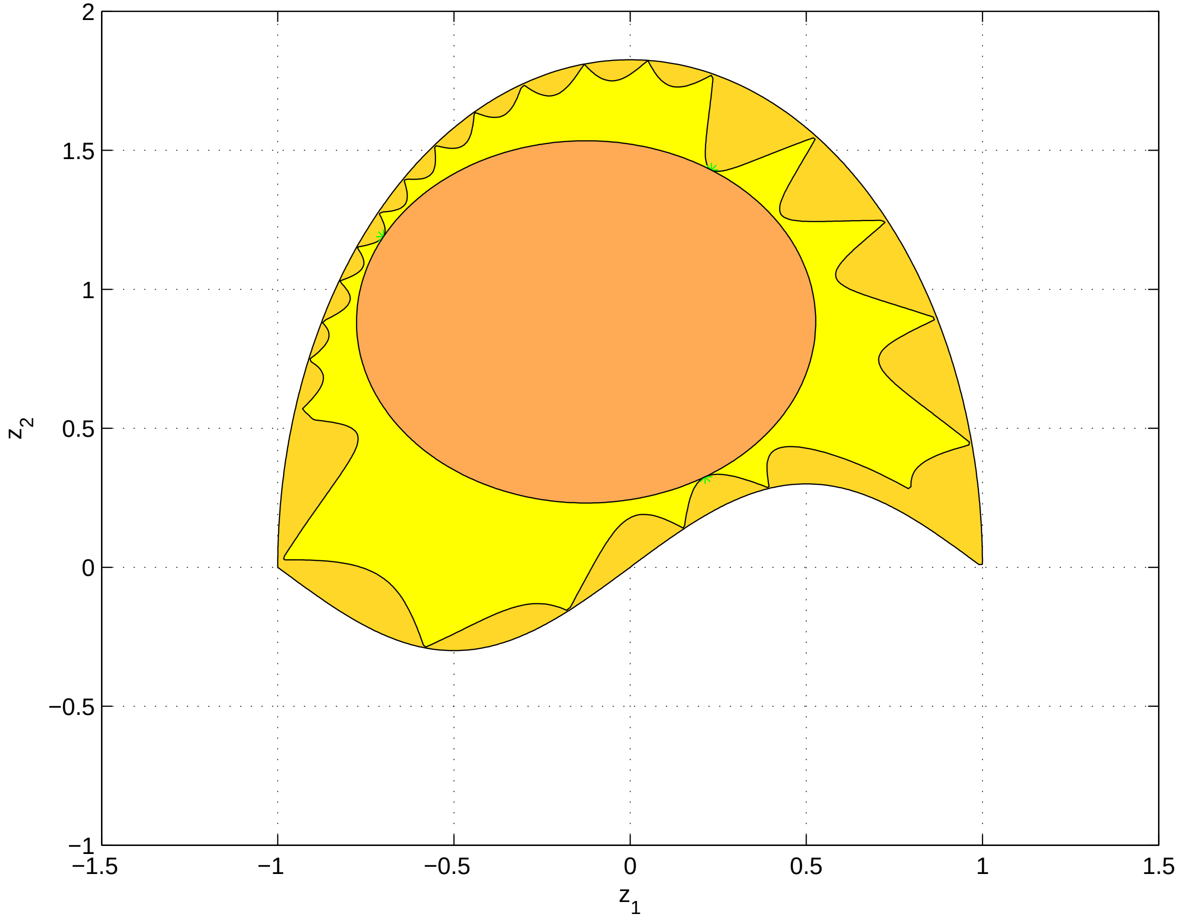
Numerical results: Design Centering



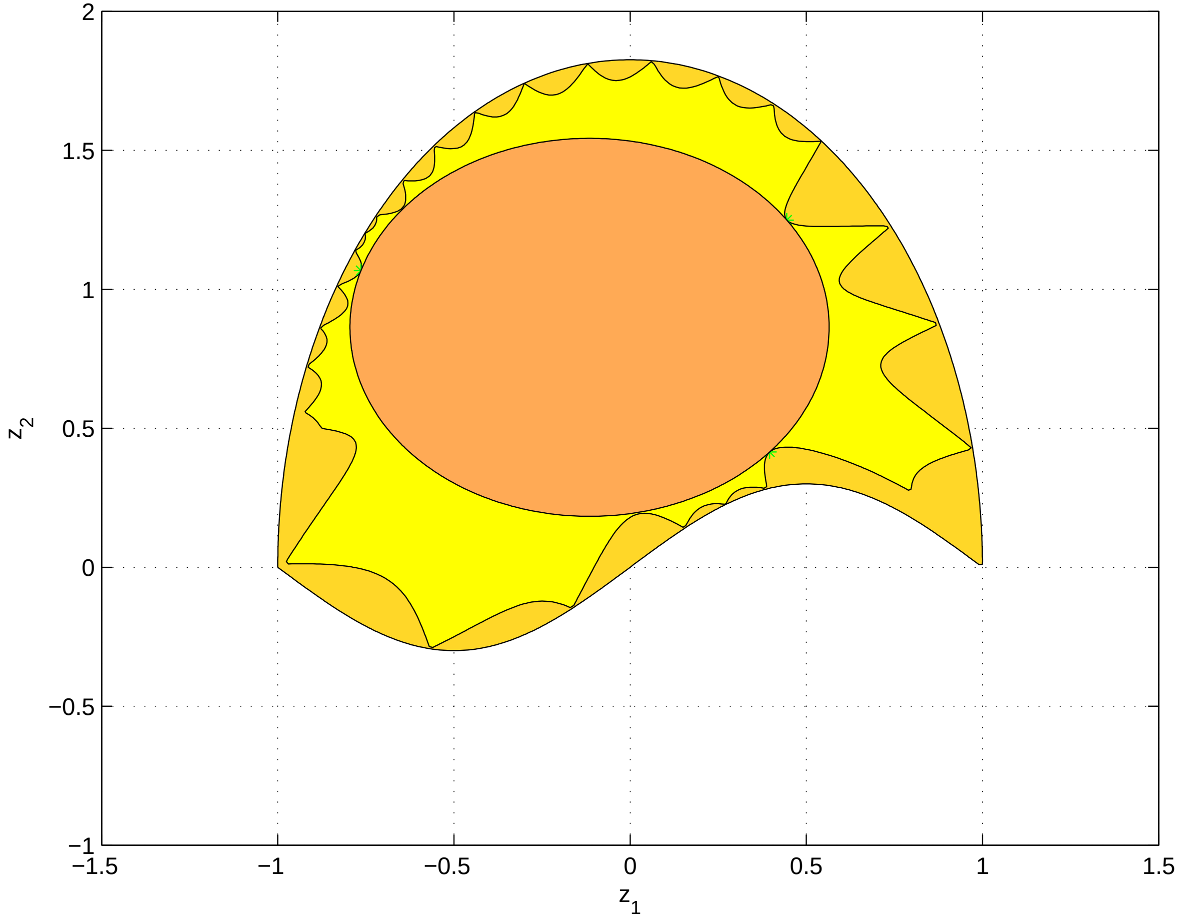
Numerical results: Design Centering



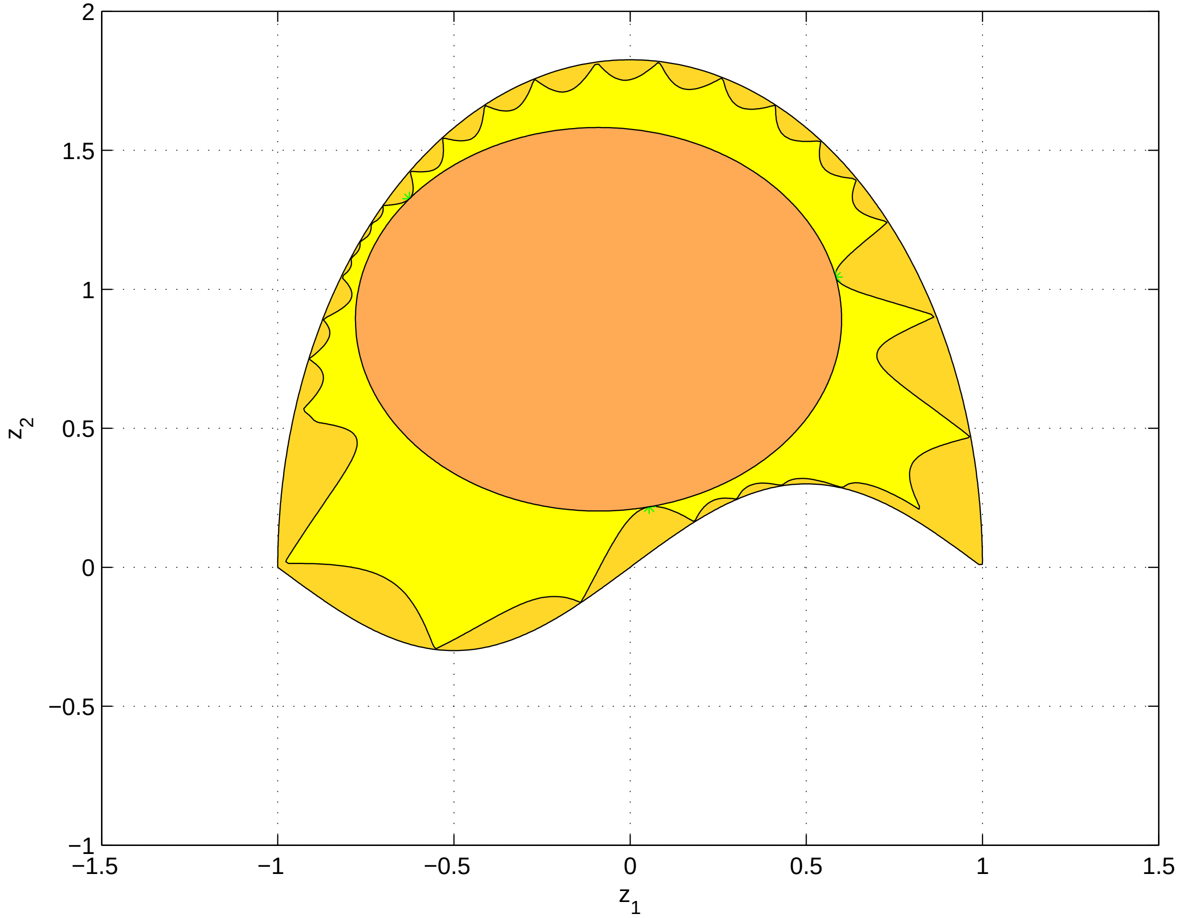
Numerical results: Design Centering



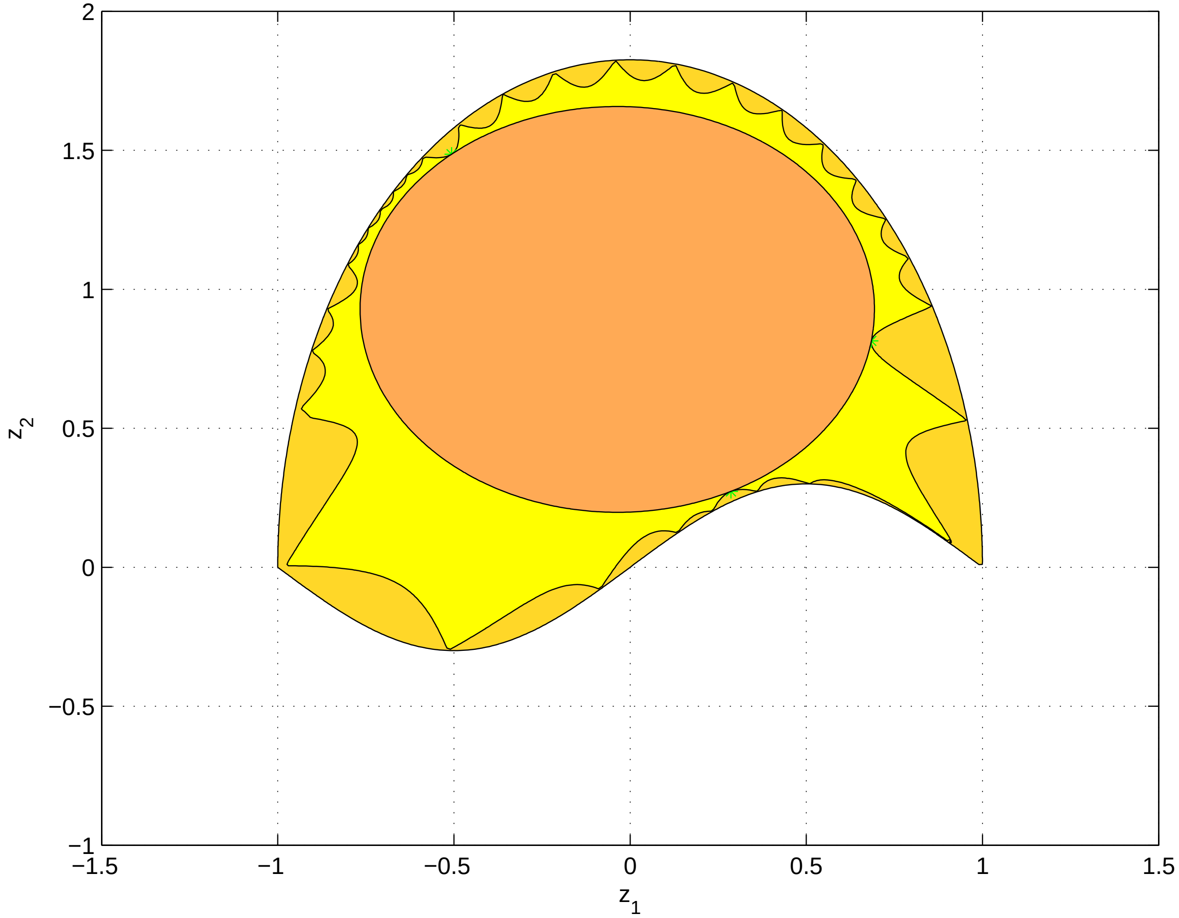
Numerical results: Design Centering



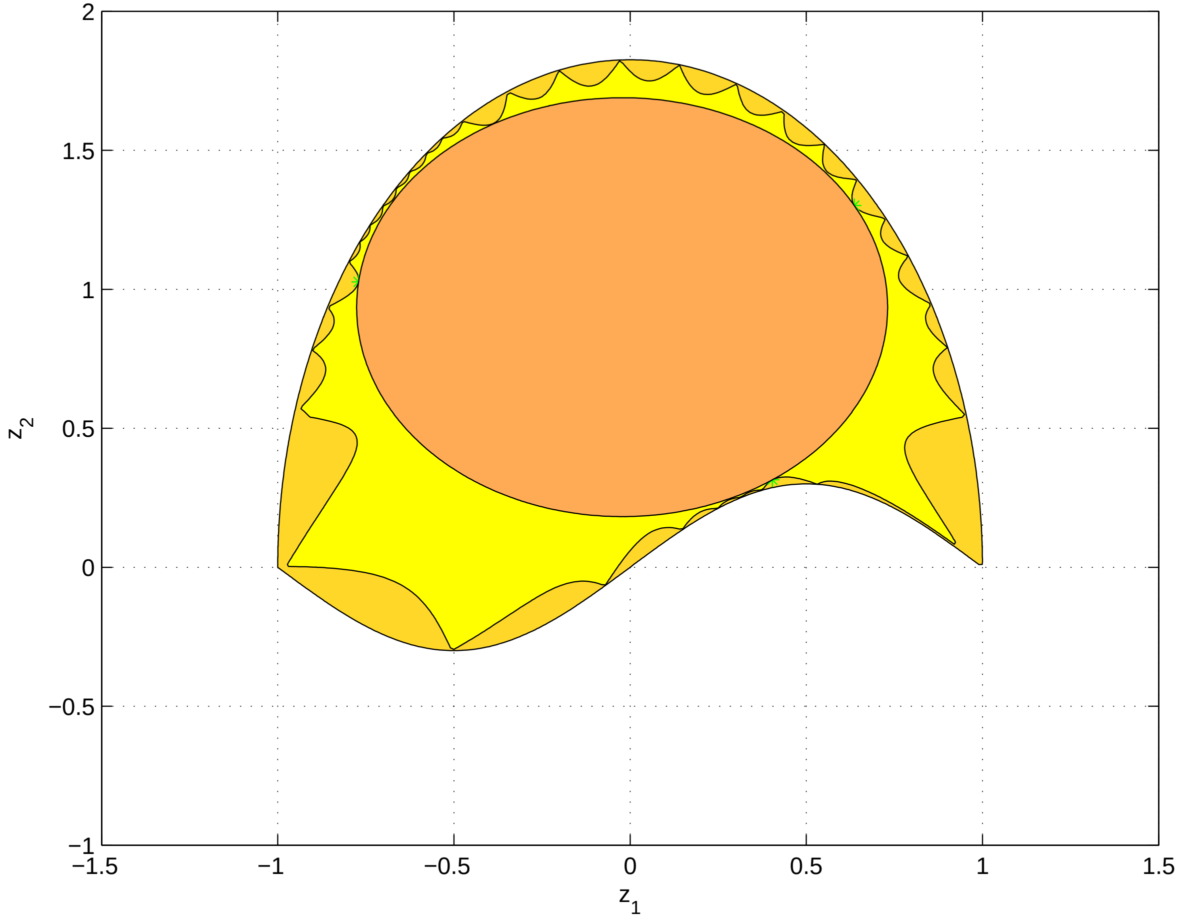
Numerical results: Design Centering



Numerical results: Design Centering



Numerical results: Design Centering



Future directions

- Generalization to $p, m \geq 1$ and non-box shaped Y
- Acceleration by better bounds and $x-$adaptation of α
- Increased stability by less restrictions
- Convergence of KKT points and local minimizers
- Interior point property of the iterates
- Application to real world Robust Optimization, Lagrange duality, cutting stock...

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Applications: Chebyshev Approximation

Approximate smooth function F on compact set Z with an element of the family $a(p,\cdot)$, $p \in P$, (e.g., polynomials), such that the *maximal* deviation is minimized:

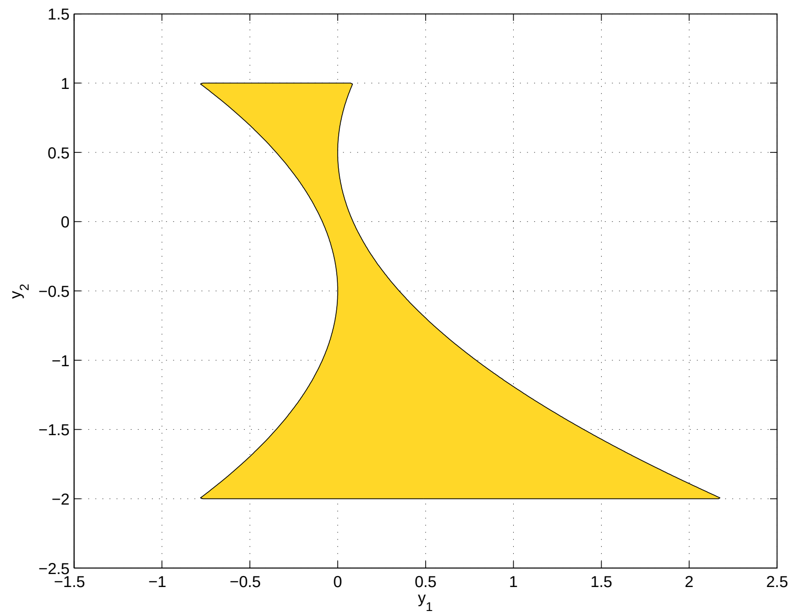
$$\min_{p \in P} ||F - a(p,\cdot)||_{\infty,Z} \;=\; \min_{p \in P} \max_{z \in Z} |F(z) - a(p,z)| \,.$$

Intrinsic nonsmoothness can be replaced by **infinitely many** smooth inequalities (*epigraph reformulation*):

$$\begin{array}{lrcl} \min_{p,q} q \quad \text{s.t.} & \max_{z \in Z} |F(z) - a(p,z)| & \leq & q \\ & & \Updownarrow & \\ & \pm(F(z) - a(p,z)) & \leq & q \quad \forall\, \boldsymbol{z} \in \boldsymbol{Z} \,. \end{array}$$

Applications: Design Centering

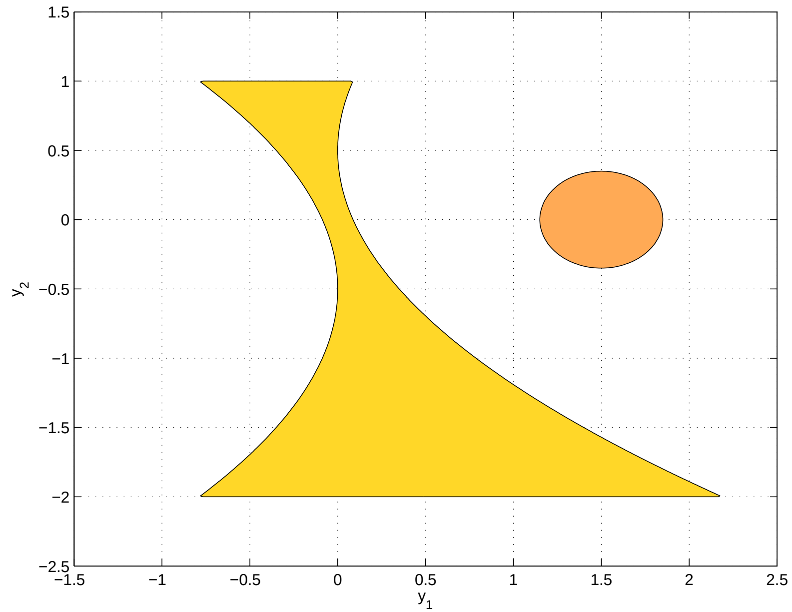
Given: a container $C \subset \mathbb{R}^m$,



Applications: Design Centering

Given: a container $C \subset \mathbb{R}^m$,

a parametrized body
$B(x) \subset \mathbb{R}^m$ with $x \in \mathbb{R}^n$.

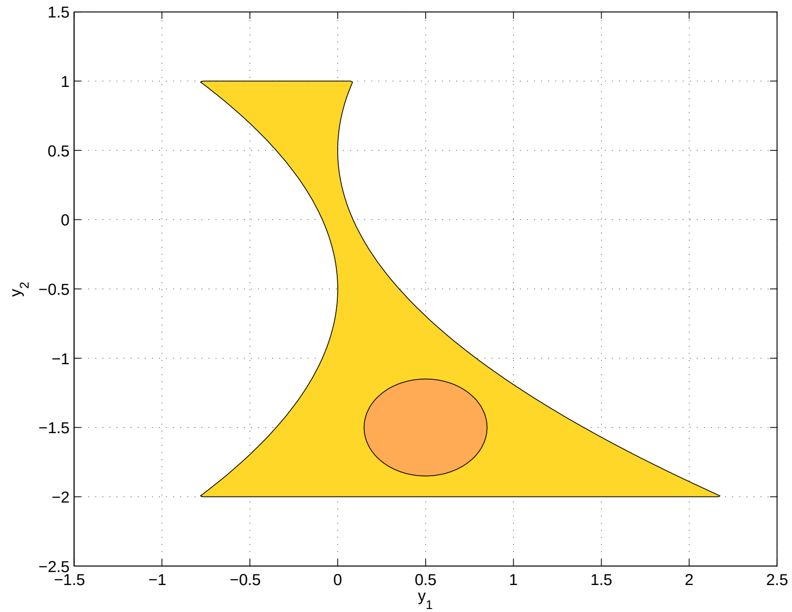


Applications: Design Centering

Given: a container $C \subset \mathbb{R}^m$,

a parametrized body $B(x) \subset \mathbb{R}^m$ with $x \in \mathbb{R}^n$.

Task: inscribe $B(x)$ into C,



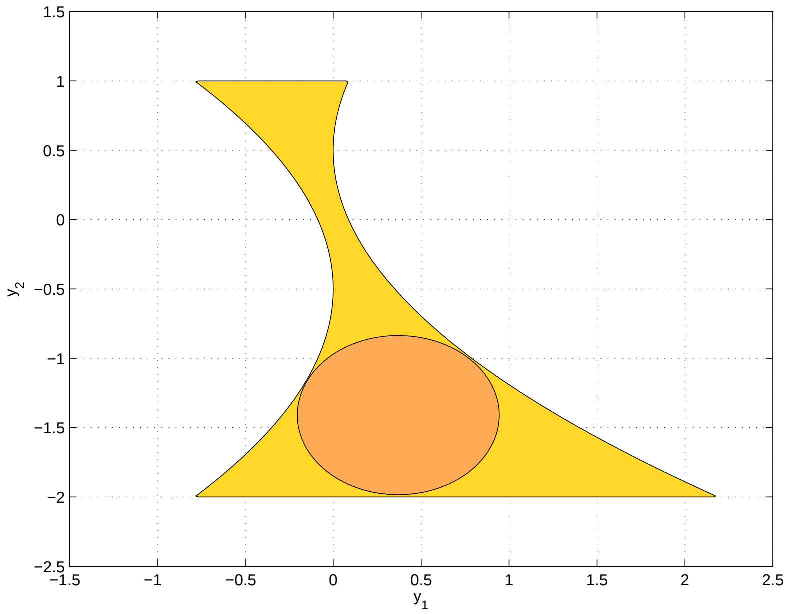
Applications: Design Centering

Given: a container $C \subset \mathbb{R}^m$,

a parametrized body $B(x) \subset \mathbb{R}^m$ with $x \in \mathbb{R}^n$.

Task: inscribe $B(x)$ into C,

such that some measure $f(x)$ of $B(x)$ is maximized.



Applications: Design Centering

Given: a container $C \subset \mathbb{R}^m$,

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$B(x) \subset \mathbb{R}^m$ with $x \in \mathbb{R}^n$.

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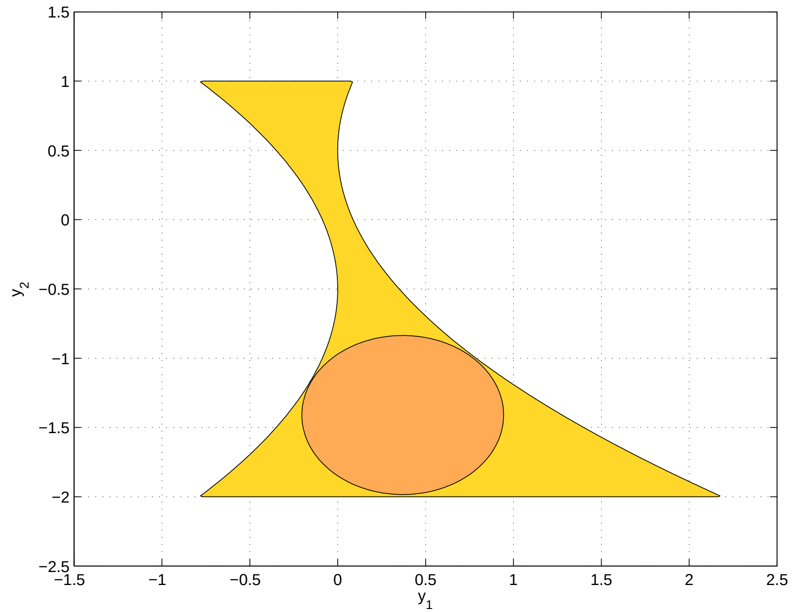

$$DC: \qquad \max_x \; f(x) \quad \text{s.t.} \quad B(x) \subset C$$

Applications: Design Centering

- Lower bounds for volume of (complicated) container C (Graettinger/Krogh 1988)

- Computation of “innermost” points (Horst/Tuy 1993)

- Model reduction (Barton e.a. 2004)

- Cutting stock (Nguyen/Strodiot 1992, Küfer/Winterfeld)