In the following an empty intersection of subsets of X will be X.

Problem 1 (6 points)**.** *For a topological space X, show the equivalence of the following conditions:*

- *We have $X \neq \emptyset$, and if U and V are non-empty open subsets of X, then $U \cap V \neq \emptyset$.*
- *If $n \in \mathbb{N}$ and $(U_i)_{i=1}^n$ are non-empty open subsets of X then $\bigcap_{i=1}^n U_i \neq \emptyset$.*
- *If $n \in \mathbb{N}$ and $(A_i)_{i=1}^n$ are proper closed subsets of X then $\bigcup_{i=1}^n A_i \neq X$.*
- *We have $X \neq \emptyset$ and every open subset of X is connected.*
- *We have $X \neq \emptyset$ and every non-empty open subset of X is dense in X.*

Definition 1. *A topological space is called* irreducible *if it satisfies these equivalent conditions. A subset of a topological space will be called irreducible if it becomes irreducible when equipped with the induced topology.*

Problem 2 (1 point)**.** *Let X be a topological space, $x \in X$ and $\overline{\{x\}}$ the closure of X. Show that $\overline{\{x\}}$ is an irreducible subset of X.*

Definition 2. *We call x a* generic point *of the irreducible closed subset $Z \subseteq X$ if $Z = \overline{\{x\}}$.*

The following separation axioms will be important in what follows:

T_0: If $x \neq y$ are points of X, then there exists a subset $M \subseteq X$ which is open or closed and such that $x \in M$ and $y \notin M$.

T_1: If $x \neq y$ are points of X, then there exists an open subset $U \subseteq X$ such that $x \in U$ and $y \notin U$.

T_2: If $x \neq y$ are points of X, then there exist disjoint open subsets U and V such that $x \in U$, $y \in V$. This is the Hausdorff axiom.

T_3: X is T_0, and if $x \in X$ and $A \subseteq X$ is a closed subset of X not containing x, then there are disjoint open subsets U and V such that $x \in U$ and $A \subseteq V$.

T_4: X is T_1, and if A and B are disjoint closed subsets of X then there are disjoint open subsets U and V of X such that $A \subseteq U$ and $B \subseteq V$.

It is rather easy to see that each of these axioms implies the earlier ones. Some of these axioms have obvious equivalent versions. For instance, T_1 is equivalent to every point being closed and T_2 to the diagonal being a closed subset of $X \times X$.

Problem 3 (1 point)**.** *Show that X is T_0 if and only if every irreducible subset has at most one generic point.*

Definition 3. *We call X* sober *if it is T_0 and every irreducible closed subset has a generic point.*

Problem 4 (3 points)**.** *Show that every finite T_0-space is sober.*

Recall that a set $\mathfrak{F}$ of subsets of a set X is called a *filter* if the following conditions hold:

- $X \in \mathfrak{F}$ and $\emptyset \notin \mathfrak{F}$.
- If $M \in \mathfrak{F}$ and $M \subseteq N \subseteq X$ then $N \in \mathfrak{F}$.
- If $M, N \in \mathfrak{F}$ then $M \cap N \in \mathfrak{F}$.

A *ultrafilter* is a $\subseteq$-maximal element of the set of filters or, equivalently, a filter such that for every $M \subseteq X$ one of M or $X \setminus M$ belongs to $\mathfrak{F}$. By Zorn's lemme every filter is contained in some ultrafilter.

If X is also equipped with a topology then we say that $x \in X$ is a *point of condensation of* $\mathfrak{F}$ if every neighbourhood of x has a non-empty intersection with every element of $\mathfrak{F}$ and a *limit of* $\mathfrak{F}$ if every neighbourhood of x belongs to $\mathfrak{F}$. Obviously every limit is a point of condensation and the opposite implication holds for ultrafilters but not for general filters.

Definition 4. *A topological space X is called* quasi-compact *it the following equivalent conditions hold:*

- *Every open covering has a finite subcovering.*
- *If $(A_i)_{i\in I}$ is a family of closed subsets and $\bigcap_{i\in F} A_i \neq \emptyset$ for all finite subsets $F \subseteq I$ then $\bigcap_{i\in I} A_i \neq \emptyset$.*
- *Every filter has a point of condensation.*
- *Every ultrafilter has a limit.*

If is called compact *if it is quasi-compact and Hausdorff.*

Problem 5 (5 points)**.** *Show that the above conditions are indeed equivalent, where the equivalence of the first two conditions should be taken for granted.*

Definition 5. *A* topology base *of a topological space X is a set $\mathfrak{B}$ of open subsets of X such that every open subset of X is a union of elements of $\mathfrak{B}$.*

It is easy to see that a set $\mathfrak{B}$ of subsets of a set X is a topology base for some topology on X if and only if every finite intersection in X (including the empty intersection which is X) is a union of elements of $\mathfrak{B}$.

Problem 6 (2 points)**.** *For a topological space X, show the equivalence of the following two conditions:*

- *There is a topolgy base $\mathfrak{B}$ closed under finite intersections in X and such that the elements of $\mathfrak{B}$ are quasicompact.*
- *The set of quasicompact open subsets of X is such a topology base.*

For a topological space X let $\mathfrak{Qc}(X)$ be the set of quasi-compact open subsets of X. Is is easy to see (and may used without further ado in all solutions) that this is closed under finite unions.

Definition 6. *A topological space is called* spectral *if it is sober and satisfies the equivalent conditions of Problem 6.*

Problem 7 (1 point)**.** *Let $Y \xrightarrow{f} X$ be a map between topological spaces, where X satfies the conditions of Problem 6. Also, let $\mathfrak{B}$ be a topology base on X as in Problem 6. Show that the following conditions are equivalent:*

- *If $\Omega \in \mathfrak{Qc}(X)$ then $f^{-1}\Omega \in \mathfrak{Qc}(Y)$.*
- *If $\Omega \in \mathfrak{B}$ then $f^{-1}\Omega \in \mathfrak{Qc}(Y)$.*

Obviously every map with these properties is continuous.

Definition 7. *A* spectral map *is a map between spectral spaces satisfying the equivalent conditions of Problem 7.*

Oviously spectral spaces form a category with spectral maps as morphisms. Every spectral map is continuous but in general there are continuous maps which are not spectral.

Problem 8 (3 points)**.** *Let $X \xrightarrow{f} Y$ be continuous where X is quasicompact and Y Hausdorff. Show that f closed in the sense that the image of a closed subset of X is closed in Y.*

Two of the 22 points from this sheet are bonus points which are not counted in the calulation of the 50%-threshold for passing the exams.

Solutions should be e-mailed to my institute e-mail address (my second name (franke) at math dot uni hyphen bonn dot de) before Monday October 28.