Introduction to C^*-algebras

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Graduate Seminar on Global Analysis
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During this seminar we will develop the basic theory of C^*-algebras. Among the main goals are two theorems by Gelfand and Naimark, which state:

1. that every commutative C^*-algebra is given by continuous functions on a (locally compact, Hausdorff) space;
2. that every C^*-algebra is isometrically isomorphic to a closed $*$-subalgebra of bounded operators on a Hilbert space.

The first statement suggests that a noncommutative C^*-algebra can be thought of as continuous functions on some 'noncommutative space'. Thus, C^*-algebras can be thought of as describing 'noncommutative topology', and as such they form the first stepping stone towards a description of 'noncommutative geometry'.

Seminar outline

The seminar talks will take place on Wednesdays, 12:00–14:00, in Seminarraum 0.006. In the first part of the seminar, we will develop the basic theory of C^*-algebras, mostly following [Ped89, Ch. 4] and [Tak01, Ch. I].

Talk 0 – 10.10.2018 by Koen van den Dungen:
Introductory overview of the seminar and planning of the talks; preliminaries on Banach spaces and linear operators.

Talk 1 – 17.10.2018 by Parthiv Basu:
Introduction to Banach algebras: ideals, unitisations, spectrum, holomorphic functional calculus. The talk should cover the material in [Ped89, §4.1]. See also [Tak01, §I.1-I.2].

Talk 2 – 26.10.2018 by Ödül Tetik:
The Gelfand transform for Banach algebras, following [Ped89, §4.2] and [Tak01, §I.3]. The talk should describe the bijection between maximal (regular) ideals and characters, construct the Gelfand transform, and show that the space of characters is locally compact and Hausdorff.

Talk 3 – 2.11.2018
After introducing the definition and basic properties of C^*-algebras, this talk discusses the Gelfand-Naimark duality between commutative C^*-algebras and locally compact Hausdorff spaces, and the continuous functional calculus. The talk should cover at least the material in [Tak01, §I.4 up to Prop 4.8] (see also [Ped89, §4.3]).

Talk 4 – 9.11.2018 by Lennart Ronge:
In this talk, several new objects are introduced, mostly following [Tak01, §I.5-I.7]. The talk should discuss at least homomorphisms between C^*-algebras [Tak01, 5.2-5.4], positive elements: uniqueness of the square root ([Ped89, 4.4.8] or [Mur90, Thm 2.2.1]) and the positive cone [Tak01, §I.6]), and approximate units [Tak01, Coro 7.5].

Talk 5 – 16.11.2018 by Marco Ronchese:
This talk introduces positive linear functionals and representations [Tak01, §I.9], and in particular describes the Gelfand-Naimark-Segal (GNS) construction and proves the existence of faithful representations for any C^*-algebra (this is the second Gelfand-Naimark theorem). The talk could follow the outline of [Str, §5 up to 5.17].

Talk 6 – 7.12.2018 by Lennart Ronge:
This talk gives a brief introduction to von Neumann algebras, following [Ped89, §4.6] and [Tak01, §II.3] (see also [Str, §5]). The talk should discuss the weak and strong operator topologies, the double commutant theorem [Ped89, 4.6.7], the fact that a von Neumann algebra is spanned by its projections [Mur90, Thm 4.1.11(1)], and the Borel functional calculus [Mur90, §4.4] (see also [Ped89, §4.5]).

Talk 7 – 14.12.2018 by Koen van den Dungen:
In this talk we consider pure states and irreducible representations, in particular [Mur90, Thms 5.1.5-7]. In the commutative case, pure states are characters. Given any C^*-algebra A, define the spectrum $\hat{A}$ as the set of unitary equivalence classes of irreducible representations, and show that there is a canonical bijection between $\hat{A}$ and the space of pure states on A. Finally, the pure states (along with the zero functional) form the extreme points of the set of norm-decreasing linear functionals [Mur90, Thm 5.1.8].

The schedule for the remaining talks is as follows.
Talk 8 – 21.12.2018
Talk 9 – 11.01.2019
Talk 10 – 18.01.2019
Talk 11 – 25.01.2019
Talk 12 – 1.02.2019

These remaining talks will be selected from the following topics:
Topic a: Discuss basic examples of C^*-algebras, such as continuous functions on topological spaces and essentially bounded functions on measure spaces. Prove that every finite-dimensional C^*-algebra is a matrix algebra [Tak01, §I.11].
Topic b: Multiplier algebras: describe the equivalence of various definitions (via concrete representation, via double centralisers, via Hilbert modules). See e.g. [Mur90, §2.1], [Lan95, Ch. 2], and [RW98, §2.3].
Topic c: Morita equivalence and stable isomorphism [RW98, §3.2&§5.5]. Prove that two σ-unital C^*-algebras are stably isomorphic if and only if they are Morita equivalent [RW98, Thm 5.55].
Topic d: Tensor products ([Mur90, §6.3] and [Tak01, §III.4]). Start with a discussion of cross-norms for tensor products of Banach spaces. For C^*-algebras, show the existence of minimal/maximal C^*-norms for tensor products of C^*-algebras [Tak01, Ch. III, §4.4].
Topic e: Group C^*-algebras [Dav96, §VII.1-2]. Describe both reduced and full group C^*-algebras. For locally compact groups, prove that the reduced and full group C^*-algebras are isomorphic if and only if the group is amenable [Dav96, Thm VII.2.5].
Topic f: Crossed products [Dav96, §VIII.1-2]. Describe the construction of a crossed product C^*-algebra from a C^*-dynamical system (given by a group acting on a C^*-algebra).
Topic g: Classification of von Neumann algebras.
Topic h: Classification of certain classes of C^*-algebras.
Topic i: Gelfand-Naimark duality revisited: show that the duality between commutative C^*-algebras and locally compact Hausdorff spaces extends to an equivalence of categories [GVF01, §1.3].

Recommended literature

[Bla06] B. Blackadar, *Operator algebras: Theory of C^*-algebras and von Neumann algebras*, Encyclopaedia of Mathematical Sciences, vol. 122, Springer, 2006.
[Dav96] K. Davidson, *C^*-algebras by example*, Fields Institute for Research in Mathematical Sciences Toronto: Fields Institute monographs, American Mathematical Soc., 1996.
[GVF01] J. Gracia-Bondía, J. Várilly, and H. Figueroa, *Elements of Noncommutative Geometry*, Birkhäuser Advanced Texts, 2001.
[Lan95] E. Lance, *Hilbert C^*-modules: A toolkit for operator algebraists*, Lecture note series: London Mathematical Society, Cambridge University Press, 1995.
[Mur90] G. Murphy, *C^*-algebras and Operator Theory*, Academic Press, 1990.
[Ped89] G. Pedersen, *Analysis now*, Graduate texts in mathematics, vol. 118, Springer-Verlag, 1989.
[RW98] I. Raeburn and D. Williams, *Morita Equivalence and Continuous-trace C^*-algebras*, Mathematical surveys and monographs, American Mathematical Society, 1998.
[Str] K. Strung, *Invitation to C^*-algebras*, available on `http://strung.me/karen/The_Fringe/invitation%20to%20C*-algebras.pdf`.
[Tak01] M. Takesaki, *Theory of operator algebra I*, Encyclopaedia of Mathematical Sciences, vol. 124, Springer, 2001.