ALGEBRA II, SHEET 8

Exercise 4

Let k be a field of positive characteristic p, and $\tau \in k$. Set $f = X^p - XY^p - \tau \in R = k[X,Y]$, and $A = R/(f)$. We have the exact sequence of A-modules

$$(f)/(f^2) \longrightarrow \Omega_{R|k} \otimes_R A \longrightarrow \Omega_{A|k} \longrightarrow 0.$$

Under the identification $\Omega_{R|k} \cong R^2$, the first map is $f \mapsto (-Y^p, 0) \in A^2$, and we deduce that

$$\Omega_{A|k} \cong A/(Y^p) \oplus A.$$

In particular, if $\mathfrak{m} \subseteq A$ is a maximal ideal such that $Y \notin \mathfrak{m}$, then $(\Omega_{A|k})_\mathfrak{m} \cong 0 \oplus A_\mathfrak{m}$ is free. Otherwise, we have $\mathfrak{m} = (Y)$. The multiplication map $A \xrightarrow{\cdot Y^p} A$ is injective, because $f \notin (Y)$. Since localization is exact, we obtain the short exact sequence

$$0 \longrightarrow A_\mathfrak{m} \xrightarrow{\cdot Y^p} A_\mathfrak{m} \longrightarrow (A/(Y^p))_\mathfrak{m} \longrightarrow 0.$$

If $(\Omega_{A|k})_\mathfrak{m}$ were free, the right-hand term would be projective. Then Y^p would have a left inverse, and thus $Y \in A_\mathfrak{m}^\times$, which is absurd. We conclude that the (open) subset of $\operatorname{Spec}(A)$ where $\Omega_{A|k}$ is locally free is precisely $\operatorname{Spec}(A) \smallsetminus \{(Y)\}$.

Let $\mathfrak{m} \subseteq A$ be any maximal ideal again. Then $A_\mathfrak{m}$ is regular if and only if $\dim(\mathfrak{m}/\mathfrak{m}^2) = 1$. Let $\mathfrak{n} \subseteq R$ be the associated maximal ideal with $\mathfrak{m} = \mathfrak{n}/(f)$. Note that $A/\mathfrak{m} = R/\mathfrak{n}$. We have a short exact sequence of vector spaces over this residue field

$$0 \longrightarrow (\mathfrak{n}^2, f)/\mathfrak{n}^2 \longrightarrow \mathfrak{n}/\mathfrak{n}^2 \longrightarrow \mathfrak{m}/\mathfrak{m}^2 \longrightarrow 0.$$

Since $\dim(\mathfrak{n}/\mathfrak{n}^2) = 2$, we deduce that $A_\mathfrak{m}$ is regular if and only if $f \notin \mathfrak{n}^2$. Now, if $(x,y) \in \overline{k}^2$ is a root of $\mathfrak{n}$, then $\mathfrak{n} = (g,h)$ is generated by the corresponding minimal polynomials over k.

If $y \neq 0$, then also $\frac{\partial f}{\partial X}(x,y) = -y^p \neq 0$, and hence $f \notin \mathfrak{n}^2 = (g^2, gh, h^2)$, so $A_\mathfrak{m}$ is regular.

If $y = 0$, then $\mathfrak{n} = (g, Y)$, and $\mathfrak{m} = (Y)$. In this case, $f \equiv X^p - \tau \bmod \mathfrak{n}^2$, and so $A_\mathfrak{m}$ is regular if and only if $X^p - \tau \notin \mathfrak{n}^2 = (g^2, gY, Y^2)$, that is, $g^2 \nmid (X^p - \tau)$.

But since $X^p - \tau = f + XY^p \in \mathfrak{n} = (g, Y)$, we see that $g \mid (X^p - \tau)$, and therefore,

$$g = \begin{cases} X^p - \tau & \text{if } \sqrt[p]{\tau} \notin k, \\ X - \sqrt[p]{\tau} & \text{otherwise.} \end{cases}$$

We conclude that A is regular if $\sqrt[p]{\tau} \notin k$, whereas otherwise, its regular locus coincides with its smooth locus $\operatorname{Spec}(A) \smallsetminus \{(Y)\}$.

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