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The Toric Variety $\mathrm{Bl}_{n+1}\mathbb{P}^n$ and its Associated Secondary Fan

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Master Thesis

submitted June 5, 2014 at
ETH Zürich

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Acknowledgement

I would like to thank my advisor Professor Brent Doran for his advice and for the helpful discussions. He gave me great freedom in my approach to the research topic and encouraged me to follow my own intuitions and interests. I also want to thank Professor Emo Welzl for an interesting discussion and a broader perspective on (regular) triangulations.

0.1. Notation

A^B	the set of functions from B to A
$\mathcal{P}(X)$	the power set of a set X
$A \dot\cup B$	the disjoint union of A and B
S_n	the symmetry group of automorphisms of the set $\{1, \ldots, n\}$
$\mathbb{N}$	the semigroup $\{0, 1, 2, 3, \ldots\}$ of natural numbers
$\mathbb{Z}$	the group $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$ of integers
$\mathbb{Q}$	the field of rational numbers
$\mathbb{R}$	the field of real numbers
$\mathbb{C}$	the field of complex numbers
$\mathbb{R}_{\geq 0}$	the set $[0, \infty) \subset \mathbb{R}$ of nonnegative real numbers (similarly $\mathbb{R}_{>0}, \mathbb{R}_{\leq 0}, \mathbb{R}_{<0}$)
$\mathrm{GL}(n, R)$	the invertible $n \times n$-matrices over a ring R
log	the natural logarithm
$\mathrm{Int}(A)$	the interior of a set A in a topological space
$V(I)$	the vanishing set of an ideal $I \subset \mathbb{C}[x_1, \ldots, x_n]$
$D(f)$	the complement of the vanishing set $V((f))$
$\mathrm{Spec}(R)$	the spectrum of a ring R
$\mathrm{Proj}(S)$	the Proj-scheme of a graded ring S

T_pX	the tangent space of a quasi-projective variety X at a non-singular point $p \in X$
$\mathrm{Cl}(X)$	the Class group of a variety X
$\mathcal{O}_X$	the structure sheaf of a variety X
$\mathcal{O}_X(D)$	the line bundle on X associated to a Cartier divisor D
$\mathbb{P}(V)$	the projective bundle associated to a vector bundle V
Bl_pX	the blow-up of a quasi-projective variety X at a smooth point $p \in X$
$\mathrm{Bl}_{n+1}\mathbb{P}^n$	the blow-up of n points in general position on $\mathbb{P}^n$
$H^0(X,V)$	the space of global sections of a vector bundle $V \to X$
$\mathrm{Span}(S)$	the linear span of a subset S of a vector space
$\mathrm{Cone}(S)$	the cone spanned by a (finite) set S in a real vector space
$\mathrm{RelInt}(\sigma)$	the interior of σ relative to the smallest affine subspace containing σ
$\sigma^\vee$	the dual cone of σ
$U^\perp$	the orthogonal complement of a linear subspace U of a Euclidean vector space
$\tau \leq \sigma$	the cone τ is a face of the cone σ
$\sigma(d)$	the set of d-dimensional faces of a cone σ
u_ρ	the generator of a ray ρ
$\Sigma(d)$	the set of d-dimensional cones in a fan Σ
$\lvert\Sigma\rvert$	the support of a fan Σ
T_n	the algebraic torus $(\mathbb{C}^*)^n$
S_σ	the affine semigroup $\sigma^\vee \cap M$
$\mathbb{C}[S]$	the semigroup algebra of an affine semigroup S
U_σ	the affine toric variety $\mathrm{Spec}(\mathbb{C}[S_\sigma])$
X_Σ	the toric variety of a fan Σ
γ_σ	the distinguished point in U_σ
$O(\sigma)$	the orbit of the point $\gamma_\sigma \in X_\Sigma$ for $\sigma \in \Sigma$
$V(\sigma)$	the closure of $O(\Sigma)$ in X_Σ
D_ρ	the torus invariant divisor on X_Σ corresponding to a ray $\rho \in \Sigma(1)$
$B(\Sigma)$	the irrelevant ideal associated to a fan Σ
$Z(\Sigma)$	the vanishing locus of $B(\Sigma)$
$\{-,0,+\}$	the set of signs
$\mathrm{MIN}(S)$	the minimal nonzero sign vectors in $S \subset \{-,0,+\}^m$
sign	the signum function on $\mathbb{R}$, which is $-$ on $(-\infty,0)$, 0 at 0 and $+$ on $(0,\infty)$
SIGN	the componentwise sign function $\mathbb{R}^m \to \{-,0,+\}^m$
$\mathrm{Dep}(V)$	the set of linear dependencies of a vector configuration V
$\mathcal{V}(V)$	the set of (signed) vectors of a vector configuration V
$\mathcal{C}(V)$	the set of (signed) circuits of a vector configuration V
$\mathrm{Val}(V)$	the set of value vectors of a vector configuration V
$\mathcal{V}^*(V)$	the set of (signed) covectors of a vector configuration V
$\mathcal{C}^*(V)$	the set of (signed) cocircuits of a vector configuration V

$\mathcal{M}(V)$	the oriented matroid of a vector configuration V
$\mathcal{M}^*$	the dual oriented matroid of an oriented matroid $\mathcal{M}$
$V \setminus v$	the vector configuration obtained by deleting v from V
$\mathrm{U} \setminus i$	the sign vector obtained by deleting the ith entry
V/w	the vector configuration obtained by contracting v in V
$G \curvearrowright X$	the group G acts algebraically on a variety X
Gx	the orbit $\{gx; g \in G\}$ of an element $x \in X$ under the action of G
G_x	the stabilizer $\{g \in G; gx = x\}$ of an element $x \in X$ under the action of G
$\widehat{G}$	the character group $\mathrm{Hom}(G, \mathbb{C}^*)$ of an algebraic group G over $\mathbb{C}$
$H^0(\mathbb{C}^r, \mathcal{L}_\chi)^G$	the invariant global sections of the bundle $\mathcal{L}_\chi \to \mathbb{C}^r$
$(\mathbb{C}^r)^{ss}_\chi$	the set of semistable points in $\mathbb{C}^r$ with respect to a character $\chi \in \widehat{G}$
$(\mathbb{C}^r)^{s}_\chi$	the set of stable points in $\mathbb{C}^r$ with respect to a character $\chi \in \widehat{G}$
T_N	the torus $N \otimes_\mathbb{Z} \mathbb{C}^* = \mathrm{Hom}(M, \mathbb{C}^*)$
C_ν	the cone in $N_\mathbb{R}$ spanned by the vectors in $\boldsymbol{\nu}$
C_β	the cone in $\widehat{G}_\mathbb{R}$ spanned by the vectors in $\boldsymbol{\beta}$
P_a	the polyhedron in $M_\mathbb{R}$ associated to a character $\chi^a \in \widehat{G}$
P_χ	the polyhedron in $\mathbb{R}^r$ associated to a character $\chi \in \widehat{G}$
$F_{i,a}$	the ith virtual facet of P_a
$F_{i,\chi}$	the ith virtual facet of P_χ
Σ_a	the normal fan of P_a
$\mathrm{SF}(\Sigma)$	the vector space of support functions on a generalized fan Σ
$\mathrm{CSF}(\Sigma)$	the set of convex support functions on a generalized fan Σ
φ_a	the convex function on C_ν defined in (3.10)
$\Sigma(a)$	the unique fan with support C_ν making φ_a strictly convex
$I_\varnothing(a)$	the set of indices i with $\varphi_a(\nu_i) > -a_i$
$C(P)$	the cone in $M_\mathbb{R} \times \mathbb{R}$ over a polyhedron $P \subset M_\mathbb{R}$
S_P	the semigroup algebra of $C(P) \cap M \times \mathbb{Z}$
$\tilde{\Gamma}^0_{\Sigma,I_\varnothing}$	the set of $a \in \gamma_\mathbb{R}^{-1}(C_\beta)$ with $\Sigma = \Sigma(a), I_\varnothing = I_\varnothing(a)$
$\tilde{\Gamma}_{\Sigma,I_\varnothing}$	the closure of $\tilde{\Gamma}^0_{\Sigma,I_\varnothing}$
$\tilde{\Sigma}_{\mathrm{GKZ}}$	the generalized fan in $\mathbb{R}^r$ formed by the cones $\tilde{\Gamma}_{\Sigma,I_\varnothing}$
$\Gamma_{\Sigma,I_\varnothing}$	the quotient of $\tilde{\Gamma}_{\Sigma,I_\varnothing}$ by its minimal face $\ker \gamma_\mathbb{R}$
Σ_{GKZ}	the secondary fan in $\widehat{G}_\mathbb{R}$ formed by the cones $\Gamma_{\Sigma,I_\varnothing}$
$\mathcal{J}$	the collection of $\boldsymbol{\beta}$-bases
$\mathrm{Cone}(\boldsymbol{\beta}_J)$	the cone spanned by $\beta_j, j \in J$
$\Sigma \wedge \Sigma'$	the coarsest common refinement of fans Σ, Σ' with the same support
C_A	the intersection of the cones $\mathrm{Cone}(\boldsymbol{\beta}_J)$ for $J \in A$
$\mathrm{ccr}(\boldsymbol{\beta})$	the coarsest common refinement of all triangulations of $\boldsymbol{\beta}$
G_ν	the graph of regular triangulations of a vector configuration $\boldsymbol{\nu}$
$G_{I_\varnothing}$	the subgraph of G_ν of chambers $\Gamma_{\Sigma',I'_\varnothing}$ with $I'_\varnothing \subset I_\varnothing$

$C_{I_\varnothing}$	the union of cones $\Gamma_{\Sigma',I'_\varnothing}$ with $I'_\varnothing \subset I_\varnothing$
$H_{I_\varnothing}$	the subgraph of G_ν of chambers chambers $\Gamma_{\Sigma',I_\varnothing}$ of the secondary fan
$S_{I_\varnothing}$	the union of chambers $\Gamma_{\Sigma',I_\varnothing}$ of the secondary fan
$\mathcal{I}_\varnothing$	the set of indices i with β_i in extremal position
$\tilde{G}_\nu$	the graph obtained from G_ν by contracting the subgraphs $H_{I_\varnothing}$
$\mathrm{C}(i,j)$	the circuits of $\boldsymbol{\beta}$ defined in Lemma 5.3
$\mathrm{U}(S)$	the cocircuits of $\boldsymbol{\beta}$ defined in Lemma 5.3
$\mathrm{U}(i)$	the cocircuits of $\boldsymbol{\beta}$ defined in Lemma 5.3
V_X	the variety corresponding to chamber type X in the sec. fan of $\mathrm{Bl}_3\mathbb{P}^2$
Σ_X	the fan corresponding to chamber type X in the sec. fan of $\mathrm{Bl}_3\mathbb{P}^2$
$c(v,\Sigma)$	the number of chambers $C \in \Sigma$ containing the vector v
$\Sigma_{\Sigma',i}$	the fan corresponding to a $\mathbb{P}^1$-bundle over X'_Σ defined in Section 5.3
$\mathcal{F}_n$	the recursive family of fans corresponding to chambers in the sec. fan of $\mathrm{Bl}_{n+1}\mathbb{P}^n$
$f(\theta(g(x)))$	the class of functions asymptotically bounded above by $f(c_1 g(x))$ and below by $f(c_2 g(x))$ for some $c_1, c_2 > 0$
$f(\Omega(g(x)))$	the class of functions asymptotically bounded below by $f(cg(x))$ for some $c > 0$
$f(O(g(x)))$	the class of functions asymptotically bounded above by $f(cg(x))$ for some $c > 0$
$\tilde{\boldsymbol{\nu}}$	the vector configuration $\nu_1,\dots,\nu_k,\nu_{n+2}\dots,\nu_{2n+2}$ corresponding to $\mathrm{Bl}_k\mathbb{P}^n$
$\tilde{\boldsymbol{\beta}}$	the Gale dual vector configuration to $\tilde{\boldsymbol{\nu}}$

1. Introduction

The following Master thesis gives a short introduction to toric varieties with the aim of presenting their representation as quotients of affine space by linear torus actions. We then proceed by looking more closely at the family $\mathrm{Bl}_{n+1}\mathbb{P}^n$ of the blow-up of $n+1$ points in general position on $\mathbb{P}^n$, their quotient representation and the other quotient varieties that can be obtained from this representation.
A toric variety V is an irreducible algebraic variety which contains an algebraic torus $T = (\mathbb{C}^*)^n$ as a Zariski-open subset in such a way that the action of T on itself extends to an algebraic action on V. From this definition one can show that many varieties appearing naturally in algebraic geometry are toric and that toric varieties can be studied using convex-geometric and combinatorial tools.
Many algebraic varieties are obtained by gluing affine varieties along open immersions. Often one has to write down very explicit and possibly complicated formulas for the corresponding transition maps. For toric varieties, however, this gluing information can be specified and visualized in a very nice and geometric way.
As an example we consider n-dimensional projective space $\mathbb{P}^n$ over $\mathbb{C}$, which can be obtained by gluing together $n+1$ copies $U_0, \dots, U_{n+1}$ of $\mathbb{C}^n$. The corresponding transition functions $\psi_{ji} : U_{ij} \to U_{ji}$ defined on sets $U_{ij} \subset U_i$ with

$$U_{ij} = \begin{cases} \{(x_1, \dots, x_n) \in U_i; x_j \neq 0\} & \text{for } i < j \\ \{(x_1, \dots, x_n) \in U_i; x_{j+1} \neq 0\} & \text{for } i > j \end{cases}$$

are then given by

$$\psi_{ji}(x_1, \dots, x_n) = \begin{cases} \left(\frac{x_1}{x_j}, \frac{x_2}{x_j}, \dots, \frac{x_i}{x_j}, \frac{1}{x_j}, \frac{x_{i+1}}{x_j}, \dots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \dots, \frac{x_n}{x_j}\right) & \text{for } i < j \\ \left(\frac{x_1}{x_{j+1}}, \frac{x_2}{x_{j+1}}, \dots, \frac{x_j}{x_{j+1}}, \frac{x_{j+2}}{x_{j+1}}, \dots, \frac{x_i}{x_{j+1}}, \frac{1}{x_{j+1}}, \frac{x_{i+1}}{x_{j+1}}, \dots, \frac{x_n}{x_{j+1}}\right) & \text{for } i > j \end{cases}.$$

Checking that these maps are compatible and that the gluing will produce a separated variety involves quite lengthy computations.
On the other hand we can also use that $\mathbb{P}^n$ is a toric variety. In Section 2.3.1 we will see that every separated, normal toric variety can be obtained by gluing affine toric varieties and that the complete information of the gluing will be given by a fan. A fan is a set of polyhedral cones in a finite dimensional real vector space, that "fit together nicely" (for a rigorous definition see Section 2.2). In the case of $\mathbb{P}^2$ the fan is given in Figure 1.1 below.

The three maximal cones $\sigma_0 = \mathrm{Cone}(e_1, e_2)$, $\sigma_1 = \mathrm{Cone}(e_0, e_2)$, $\sigma_2 = \mathrm{Cone}(e_0, e_1)$ appearing in the fan, will correspond to the three affine charts U_0, U_1, U_2 we saw

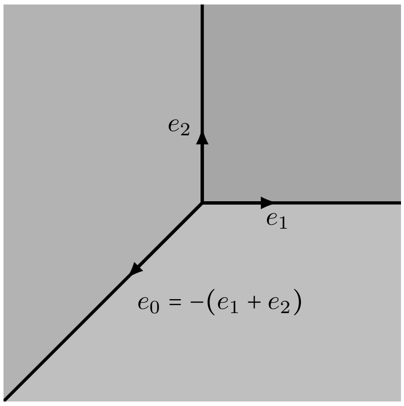


Figure 1.1.: The fan of the variety $\mathbb{P}^2$

above. The cone $\rho_2 = \text{Cone}(e_2)$, which sits inside σ_0 and σ_1 will be related to the sets $U_{01} \cong U_{10}$ above and the inclusions $\rho_2 \subset \sigma_0, \rho_2 \subset \sigma_1$ will give us open immersions $U_{01} \hookrightarrow U_0$ and $U_{10} \hookrightarrow U_1$ along which we can glue U_0 and U_1.
Many properties of a toric variety X_Σ can be checked using the fan Σ defining them. For instance the fact that $\mathbb{P}^2$ is compact in the complex topology follows from the fact that its fan "fills" the entire space $\mathbb{R}^2$ in which it lives, that is the union of all the cones is $\mathbb{R}^2$. This makes toric varieties a very accessible branch of algebraic geometry.
Another very elegant way to define the variety $\mathbb{P}^n$ is to consider the action of $\mathbb{C}^*$ on $\mathbb{C}^{n+1}$ by

$$t(x_0, \dots, x_n) = (tx_0, \dots, tx_n)$$

for $t \in \mathbb{C}^*, x_0, \dots, x_n \in \mathbb{C}$. Here, in order to have a nice structure of a variety on the quotient, we have to remove the origin from $\mathbb{C}^{n+1}$ and obtain $\mathbb{P}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$. In Section 2.3.4 we will see that every toric variety X_Σ can be obtained as the quotient of affine space by the action of a torus and we will see how to obtain this action directly from the fan Σ.
In Section 5.1.1 we will investigate the above quotient description in detail. We will see that removing the origin is essentially the only possibility to obtain a nice structure on the quotient. However, for more complicated examples (like $\text{Bl}_{n+1}\mathbb{P}^n$) we will have choices which sets to remove, and different choices will lead to different quotient varieties. The information about the possible quotients and how they are related can be obtained from another fan, called the secondary fan. Its cones correspond to the different ways to form a quotient variety and this fan will be our main object of study.

We now give an overview on the structure of the thesis.
In Chapter 2 we present the prerequisites for the later parts of the Master thesis. First we will give a reference for the required algebraic geometry. Following a brief account of the necessary convex geometry we will define toric varieties and we treat some of

their properties with a view on their quotient representation. Finally, we will introduce Gale duality, a concept of linear algebra widely used throughout the thesis, and give an overview of vector configurations and oriented matroids. They will be used in the later parts of the thesis when dealing with the geometry of the secondary fan.
The theory of toric group actions and the definition of the secondary fan will be covered in Chapter 3. We begin with a brief reminder of Geometric Invariant Theory, but already in the setting for linear torus actions on affine space. We will proceed by showing the connection between the algebraic-geometric quotient operation and the corresponding convex-geometric constructions giving us the fan of the quotient variety. Finally we define the secondary fan, show some of its properties and see how to compute it.
In Chapter 4 we analyse the geometry of the secondary fan. In the first part of the chapter we cite results which show how the quotients coming from the different cones of the secondary fan are related. Then we will use results from Chapter 3 to investigate the global structure of the secondary fan.
Finally we will cover a specific family of examples, the varieties $\mathrm{Bl}_{n+1}\mathbb{P}^n$, in greater detail in Chapter 5. We will construct their fan and quotient description and then cover the cases $n = 1, 2, 3$ in great detail. We finish by investigating the secondary fan for general n, where we find a family of chambers corresponding to quotients which we identify, and give asymptotic bounds for the number of chambers in this family and the total number of chambers in the secondary fan. As a last point we will explain why considering the blow-up of $n + 1$ points is sufficient for also understanding the blow-up of less points on $\mathbb{P}^n$.

Large parts of this thesis are based on the book [CLS11] "Toric Varieties" by Cox, Little and Schenck. These include, but are not limited to, Sections 2.2 - 2.4, Chapter 3 and the beginning of Chapter 4. One of our goals was to make the quotient description of toric varieties from the book accessible to an audience not yet familiar with toric varieties, without having to work through large parts of [CLS11].
At the beginning of each chapter we give an overview of the results presented there and references for sources that were used. Results that have been taken from a specific source or that appear there (possibly in a different phrasing or a stronger/weaker version) are designated accordingly.
To conclude we give an accout of the results that were not taken from other sources but formulated and proved by the author. Here we do not claim that these results appear nowhere else in the literature or that they are unknown to the people working actively in the corresponding fields.
In Section 3.2.2 we expand a different approach for obtaining the fan of our quotient variety, which was mentioned in [CLS11]. In the following sections we use this approach to present alternative proofs for some of the results in [CLS11]. Moreover we give an explicit description and construction for the secondary fan in 3.4.3 that was only mentioned in the primary sources of the thesis.
In Section 4.2 we use a result in [DLRS10] from the theory of triangulations to identify in a more global way how the walls in the secondary fan relate to the quotients for chambers sharing a common wall. This result inspires some of the statements about

the geometry of the secondary fan, which we present in Section 4.3. The content of this section is probably known in the area of convex geometry and triangulations, but was not treated in the context of toric varieties, at least not in our sources.
Finally to the best knowledge of the author, the vector configuration associated to $\mathrm{Bl}_{n+1}\mathbb{P}^n$ has not been studied in detail so far. Therefore the content of Sections 5.1.3 - 5.3.2 in Chapter 5, unless otherwise stated there, consists of original results.

2. Prerequisites

In the following sections some basic concepts and notation is introduced, which is needed for later chapters. All of these results are well-known and have been taken from the sources cited at the beginning of the sections or when the result is presented. As the amount of mathematical constructions and objects covered here is quite extensive, we will only **highlight** them when they are first defined, rather than using Propositions, theorems etc. as in the subsequent chapters.

2.1. Algebraic geometry

This thesis is written for a reader familiar with the basics of algebraic geometry. In particular, we will use the following notions without further explanation

- affine, projective and quasi-projective varieties and morphisms between them
- vector and fibre bundles, locally free sheaves
- line bundles, the Class group, the Picard group and ampleness
- affine varieties as the spectrum of their coordinate ring

Texts that cover this material are for example [Har77] by Hartshorne and the book [Šaf94a] by Šafarevič.
In the following we will remind the reader of two basic operations in algebraic geometry that will occur in the later sections of the thesis.

2.1.1. Blow-up of points on a variety

Blowing up points or subvarieties is a fundamental operation in birational geometry. It has many applications, for instance in the resolution of singularities. For a quasi-projective algebraic variety X and a nonsingular point $p \in X$ we want to define the **blow-up of X at p**, which is a new quasi-projective variety $\mathrm{Bl}_p X$ together with a birational morphism $\mathrm{Bl}_p X \to X$. We proceed in several steps (for more details see for instance [Har77], I.4).

- $X = \mathbb{C}^n, p = 0$: Let $x_1, \dots, x_n$ be coordinates on $\mathbb{C}^n$ and $y_1, \dots, y_n$ be (homogeneous) coordinates on $\mathbb{P}^{n-1}$. We define the blow-up of $\mathbb{C}^n$ at the origin to be the variety
$$\mathrm{Bl}_0\mathbb{C}^n = V(x_i y_j - x_j y_i; i, j = 1, \dots, n) \subset \mathbb{C}^n \times \mathbb{P}^{n-1}$$

and the map $\varphi : \mathrm{Bl}_0\mathbb{C}^n \to \mathbb{C}^n$ to be the projection on the first factor. Note that restricted to the preimage of $\mathbb{C}^n \setminus \{0\}$ this map is an isomorphism, so it is indeed birational. The preimage of 0 is exactly $\{0\} \times \mathbb{P}^{n-1}$, so we can think of the blow-up as removing 0 and replacing it with the projectivized tangent space $T_0\mathbb{C}^n$ of $\mathbb{C}^n$ at 0. This allows us to distinguish "directions through the origin" and this is one of the reasons why blowing-up can be used to resolve singularities.

- $X = \mathbb{C}^n, p \in \mathbb{C}^n$ arbitrary: Translate p to the origin by the automorphism $x \mapsto x - p$ of $\mathbb{C}^n$, repeat the construction above and translate back.
- $X \subset \mathbb{C}^n$ an affine variety, $p \in X$: Let $\varphi : \mathrm{Bl}_p\mathbb{C}^n \to \mathbb{C}^n$ be the blow-up of $\mathbb{C}^n$ at p, then we define
$$\mathrm{Bl}_p X = \overline{\varphi^{-1}(X \setminus \{p\})} \subset \mathbb{C}^n \times \mathbb{P}^{n-1},$$
where the closure is taken in the Zariski topology. The corresponding birational morphism is the restriction $\varphi|_{\mathrm{Bl}_p X}$.
- X a quasi-projective variety, $p \in X$: Let $U \subset X$ be an affine neighbourhood of p and choose an embedding $U \subset \mathbb{C}^n$. Then we obtain the blow-up of X at p by glueing the varieties $\mathrm{Bl}_p U$ and $X \setminus \{p\}$ along the open subvariety $U \setminus \{p\}$. Similarly we glue the corresponding morphisms $\mathrm{Bl}_p U \to U$ and $X \setminus \{p\} \to X$. One checks that this is well-defined and gives a quasi-projective variety $\mathrm{Bl}_p X$ together with a birational morphism $\mathrm{Bl}_p X \to X$.

We remark that the preimage of p under the map $\mathrm{Bl}_p X$ is called the **exceptional divisor** of the blow-up. From the construction above we see that indeed it is a divisor and isomorphic to $\mathbb{P}^{n-1}$.

2.1.2. Flips

An other type of operation on algebraic varieties, that we will encounter, is a flip. This is a special type of birational transformation with exceptional set of codimension ≥ 2. Flips were discovered as a step of Mori's minimal model program. We will not use the definition or the properties of flips in this thesis, but we wanted to give references for the interested reader. For a short elementary overview see [Cor04] by Corti. For a good introduction see [Kol91] by Kollár.

2.2. Convex geometry

In this section we introduce some notation from convex geometry. This introduction is based on sections 1.2 (cones), 2.2 (polytopes), 2.3 (normal fan), 3.1 (fans) and 7.1 (polyhedra) of [CLS11]. We will first introduce cones thoroughly and present the more general polyhedra later. Many constructions are defined in both contexts but as cones will be very important throughout the thesis, we first cover them in greater detail.
Let N be a lattice of rank n, i.e. a free abelian group of finite rank, and $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R} \supset N$ the corresponding real finite-dimensional vector space. Let $M, M_{\mathbb{R}}$ be the dual

lattice and vector space and denote $\langle \cdot, \cdot \rangle$ the corresponding bilinear forms. A **convex polyhedral cone** σ in $N_{\mathbb{R}}$ is a set of the form

$$\sigma = \mathrm{Cone}(S) = \left\{ \sum_{s \in S} \lambda_s s; \lambda_s \geq 0 \right\},$$

where $S \subset N_{\mathbb{R}}$ is finite. We will omit the word "convex" in the future. The **dimension** of the cone σ is the dimension of the smallest subspace L of $N_{\mathbb{R}}$ containing σ. The interior of σ seen as a subset of L is called the **relative interior** $\mathrm{RelInt}(\sigma)$ of σ. You can show that if σ is generated by the finite set $S \subset N_{\mathbb{R}}$, its relative interior has the form

$$\mathrm{RelInt}(\sigma) = \left\{ \sum_{s \in S} \lambda_s s; \lambda_s > 0 \right\}.$$

The cone σ is called **rational** if we can choose $S \subset N$ and **strongly convex** if $\sigma \cap (-\sigma) = \{0\}$.
Now we define the **dual cone**

$$\sigma^{\vee} = \{ m \in M_{\mathbb{R}}; \langle m, u \rangle \geq 0 \text{ for all } u \in \sigma \}.$$

One can show that it is again a polyhedral cone and if σ was rational, $\sigma^{\vee}$ is rational too.
Moreover, we have $(\sigma^{\vee})^{\vee} = \sigma$. From this you can show an alternative characterisation of polyhedral cones: for $\mathcal{M} = \{m_1, \ldots, m_k\} \subset M_{\mathbb{R}}$ we define the set

$$\sigma = \{ u \in N_{\mathbb{R}}; \langle m_i, u \rangle \geq 0 \text{ for } i = 1, \ldots, k \}.$$

Then $\sigma = \mathrm{Cone}(\mathcal{M})^{\vee}$ is a polyhedral cone. We call $\mathcal{M}$ the set of defining inequalities for σ. Conversely, starting with a polyhedral cone $\sigma \subset N_{\mathbb{R}}$ and choosing $\mathcal{M}$ to be a generating set for $\sigma^{\vee}$, the above construction recovers $\sigma = (\sigma^{\vee})^{\vee}$. Thus polyhedral cones are exactly the sets obtained as the intersection of a finite set of closed half-spaces.
A **face** τ of the cone σ is a set of the form $\tau = \sigma \cap \mathrm{Span}(m)^{\perp}$ for $m \in \sigma^{\vee}$. We write $\tau \preceq \sigma$. One can show that all faces are again cones and we denote by $\sigma(d)$ the set of d-dimensional faces of σ. We remark that "$\preceq$" defines a transitive relation, i.e. $\tau_1 \preceq \tau_2$ and $\tau_2 \preceq \tau_3$ implies $\tau_1 \preceq \tau_3$. A face of codimension 1 in σ is called a **facet**. For a 1-dimensional strongly convex rational polyhedral cone ρ (which we will call a **ray**), there is a unique minimal element $u_\rho \in \rho \cap N$ which generates $\rho \cap N$, i.e. $\mathbb{N} u_\rho = \rho \cap N$. It is called a **minimal generator** of ρ. A cone σ is called **simplicial** if the minimal generators u_ρ for $\rho \in \sigma(1)$ are linearly independent over $\mathbb{R}$ and it is called **smooth** if they are even a $\mathbb{Z}$-basis of N.
One of our main objects of study in the later chapters will be fans. A **fan** Σ in $N_{\mathbb{R}}$ is a finite collection of strongly convex polyhedral cones $\sigma \subset N_{\mathbb{R}}$ such that for all $\sigma, \sigma' \in \Sigma$ and $\tau \preceq \sigma$ we have $\tau \in \Sigma$ and $\sigma \cap \sigma'$ is a face of σ and σ', hence an element of Σ. If we leave out the assumption that all cones are strongly convex, we get the definition of a **generalized fan**. Similar to above we denote by $\Sigma(r)$ the set of r-dimensional cones

in Σ.
A fan $\Sigma \subset N_{\mathbb{R}}$ is called **full-dimensional** if the maximal dimension of cones in Σ is equal to the dimension n of $N_{\mathbb{R}}$. For such a fan, a cone of maximal dimension n in Σ is called a **chamber** of Σ. A cone $\tau \in \Sigma(n-1)$ is called a **wall** if it is the intersection of two chambers $\sigma, \sigma' \in \Sigma(n)$.
A fan Σ is called simplicial or smooth, respectively, if all its cones have this property. The **support** of Σ is $|\Sigma| = \bigcup_{\sigma\in\Sigma} \sigma$ and Σ is called **complete** if $|\Sigma| = N_{\mathbb{R}}$.
We will see that cones enable us to define affine toric varieties and fans will give general varieties by a glueing construction. One possibility to obtain a fan is to start with a polytope or more generally, a polyhedron. A **polytope** $P \subset M_{\mathbb{R}}$ is the convex hull of a finite set of points. Generalizing cones and polytopes we define a **polyhedron** to be the intersection of a finite number of closed affine halfspaces, so a polyhedron $P \subset M_{\mathbb{R}}$ can be described as

$$P = \{m \in M_{\mathbb{R}}; \langle m, \nu_i\rangle \geq a_i, i = 1, \ldots, r\}$$

for $\nu_1, \ldots, \nu_r \in N_{\mathbb{R}}$ and $a_1, \ldots, a_r \in \mathbb{R}$. The **dimension** of the polytope is the dimension of the smallest affine subspace containing it. A **face** $\tau \preceq P$ is a set of the form

$$\tau = P \cap \{m \in M_{\mathbb{R}}; \langle m, \nu\rangle = a\}$$

for $\nu \in N_{\mathbb{R}}$ and $a \in \mathbb{R}$ such that $\langle m, \nu\rangle \geq a$ for all $m \in P$. So for $\nu \neq 0$ we have that τ is exactly the intersection of P with an affine hyperplane H of $M_{\mathbb{R}}$ such that P lies completely in one of the closed halfspaces determined by H. This shows that it is again a polyhedron. The faces with dimension $\dim(P) - 1$ are called **facets** of the polyhedron and the zero-dimensional faces are its **vertices**.
A (nonempty) polyhedron P can always be expressed as the Minkowski sum $P = Q + C$ of a polytope Q and a cone C. This cone C is called the **recession cone** of P. The polyhedron P is called a **lattice polyhedron** if all vertices of Q are in M and C is a strongly convex rational polyhedral cone.
Now we can define the **normal fan** of a polyhedron $P \subset M_{\mathbb{R}}$ with strongly convex rational recession cone. For every vertex v of P we define $C_v = \mathrm{Cone}(P \cap M_{\mathbb{Q}} - v)$, where $M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q} \subset M_{\mathbb{R}}$. Then the dual cones $C_v^\vee$ and their faces, where v are the vertices of P, form a generalized fan Σ_P in $N_{\mathbb{R}}$. We have that its support is exactly $C^\vee$, the dual of the recession cone of P.
A polyhedron $P \subset M_{\mathbb{R}}$ of dimension d is called **simple** if every vertex v of P lies in exactly d facets of P.
We now give some results needed in later proofs. The first is a famous result by Carathéodory.

Proposition 2.1 (Carathéodory's theorem)**.** Let $X \subset \mathbb{R}^d$ finite and $x \in \mathbb{R}^d$. Then if $x \in \mathrm{Cone}(X)$ there is a subset $X' \subset X$ of at most $\dim(\mathrm{Cone}(X))$ vectors in X such that $x \in \mathrm{Cone}(X')$.

Proof. See Proposition 1.15 in [Zie95]. □

The final results in this section are well-known and elementary, but for lack of a good reference, we prove them ourselves.

Proposition 2.2. Let $\sigma \subset N_{\mathbb{R}}$ be a cone defined by a finite set of inequalities $\langle m, \cdot \rangle \geq 0$ for $m \in \mathcal{M} \subset M_{\mathbb{R}}$. Then the faces τ of σ are exactly the subsets of σ obtained by requiring some of the inequalities $\langle m, \cdot \rangle \geq 0$ in $\mathcal{M}$ to become equalities, i.e. sets of the form

$$\tau = \{u \in N_{\mathbb{R}}; \langle m, u \rangle \geq 0 \text{ for } m \in \mathcal{M}, \langle m, u \rangle = 0 \text{ for } m' \in \mathcal{M}_0\},$$

where $\mathcal{M}_0 \subset \mathcal{M}$.

Proof. As $\sigma = \text{Cone}(\mathcal{M})^\vee$ we have $\sigma^\vee = (\text{Cone}(\mathcal{M})^\vee)^\vee = \text{Cone}(\mathcal{M})$. The faces of σ are exactly the sets of the form $\tau = \sigma \cap \text{Span}(m)^\perp$ for $m \in \sigma^\vee$, i.e. $m = \sum_{m' \in \mathcal{M}} \lambda_{m'} m'$ for $\lambda_{m'} \geq 0$. But then for $u \in \sigma$ we have

$$\langle m, u \rangle = 0 \iff \langle m', u \rangle = 0 \text{ for all } m' \in \mathcal{M} \text{ with } \lambda_{m'} > 0.$$

This implies that all faces of σ are obtained by requiring a subset $\mathcal{M}_0$ of the inequalities in $\mathcal{M}$ to become equalities. Conversely, for $\mathcal{M}_0 \subset \mathcal{M}$ choosing $m = \sum_{m' \in \mathcal{M}_0} m'$ we obtain the other direction. □

Corollary 2.3. Let $\sigma, \sigma' \subset N_{\mathbb{R}}$ be polyhedral cones and let $\tau_0 \leq \sigma \cap \sigma'$ be a face of the cone $\sigma \cap \sigma'$. Then there are faces $\tau \leq \sigma, \tau' \leq \sigma'$ with $\tau_0 = \tau \cap \tau'$.

Proof. If $\mathcal{M}, \mathcal{M}' \subset M_{\mathbb{R}}$ are defining inequalities for σ, σ', the set $\mathcal{M} \cup \mathcal{M}' \subset M_{\mathbb{R}}$ is a set of defining inequalities for $\sigma \cap \sigma'$. The face τ_0 is obtained by requiring a subset $\mathcal{M}_0 \subset \mathcal{M} \cup \mathcal{M}'$ of these inequalities to become equalities. Then the faces $\tau \leq \sigma, \tau' \leq \sigma'$ obtained by requiring equality for $\mathcal{M} \cap \mathcal{M}_0, \mathcal{M}' \cap \mathcal{M}_0$ have the desired property. □

2.3. Toric varieties

We now give a very brief introduction into the field of toric varieties. The following sections are based on Chapters 1 and 3 (definition of toric varieties, examples), 4 (divisors), 5 (toric varieties as quotients) and 7 (projective bundles) of [CLS11]. We mostly follow the notation of the book.
First we define the toric variety of a fan and relate some combinatorial properties of the fan with the geometric properties of the corresponding variety. We discuss the orbit-cone correspondence and use it to define torus-invariant divisors. This leads to the computation of the Picard group, which already hints to the construction of a toric variety as a good categorical quotient in Section 2.3.4.
In the following we are going to work over the complex numbers. The algebraic **torus** T_n of rank n is a finite product $(\mathbb{C}^*)^n$ together with the canonical product structure of varieties and the group structure obtained by componentwise multiplication. A **toric variety** V is an irreducible algebraic variety which contains a torus T_n as a Zariski-open subset in such a way that the action of T_n on itself by (left-)translation extends to an algebraic action on V. Toric varieties are a very interesting field of study and although the definition may seem rather special at first, many of the standard examples in algebraic geometry are toric. We will now see several methods of constructing toric varieties from combinatorial objects.

2.3.1. The toric variety of a fan

Let $\sigma \subset N_{\mathbb{R}}$ be a polyhedral cone and $\sigma^{\vee}$ its dual cone. It has full dimension n if and only if σ is strongly convex. Consider

$$S_{\sigma} = \sigma^{\vee} \cap M.$$

It carries the structure of a semigroup and in fact, it is finitely generated, abelian and sits inside the lattice M. Such a semigroup is called an **affine semigroup**. For any affine semigroup S we define its **semigroup algebra** $\mathbb{C}[S]$ to be the vector space over S (call the corresponding basis elements $\chi^m, m \in S$) with multiplication operation induced by $\chi^m \cdot \chi^{m'} = \chi^{m+m'}$. Then one can show that $\mathbb{C}[S]$ is a finitely generated integral domain. Hence $\operatorname{Spec}(\mathbb{C}[S])$ is an irreducible affine variety.
For affine semigroups $S \hookrightarrow S'$ we have a corresponding inclusion $\mathbb{C}[S] \hookrightarrow \mathbb{C}[S']$ and induced dominant morphism $\operatorname{Spec}(\mathbb{C}[S']) \to \operatorname{Spec}(\mathbb{C}[S])$. In particular, the inclusion $S_{\sigma} \hookrightarrow M$ induces a map $\operatorname{Spec}(\mathbb{C}[M]) \to \operatorname{Spec}(\mathbb{C}[S_{\sigma}])$ and it is easy to see that $\mathbb{C}[M]$ can be identified with the Laurent polynomials in $t_1, \dots, t_n$ (where t_i^r corresponds to χ^{re_i} with $e_1, \dots, e_n$ a basis of M). Hence $T_n = \operatorname{Spec}(\mathbb{C}[M]) = (\mathbb{C}^*)^n$ is a torus and one can show that the map above defines an inclusion which makes $\operatorname{Spec}(\mathbb{C}[S])$ an affine toric variety. Additionally, all such varieties are of this form. The affine toric variety obtained from $S_{\sigma} = \sigma^{\vee} \cap M$ for a strongly convex polyhedral cone $\sigma \subset N_{\mathbb{R}}$ is designated by U_{σ}. As a first example, note that for $\sigma = \{0\} \subset N_{\mathbb{R}}$ we have $\sigma^{\vee} = M_{\mathbb{R}}$ and thus $S_{\sigma} = M$. Hence we can construct the torus T_n as the affine toric variety $U_{\{0\}}$.
Returning to the general setting note that we can identify the torus T_n with $T_N = N \otimes_{\mathbb{Z}} \mathbb{C}^*$ by choosing a basis of N and using $\mathbb{Z}^n \otimes_{\mathbb{Z}} \mathbb{C}^* = (\mathbb{C}^*)^n$. Observe that the definition of T_N does not depend on such a choice so in many situations we will use this more intrinsic definition. Note too that we can intrinsically identify the torus $\operatorname{Spec}(\mathbb{C}[M])$ with $T_N = \operatorname{Hom}(M, \mathbb{C}^*)$, because a homomorphism $\gamma : M \to \mathbb{C}^*$ extends to a surjective ring map $\mathbb{C}[M] \to \mathbb{C}$ where $\chi^m \mapsto \gamma(m)$ and the kernel of this map is a maximal ideal, corresponding to a point in T_N.
Given a fan Σ in $N_{\mathbb{R}}$ we define a toric variety X_{Σ} as follows: every $\sigma \in \Sigma$ gives an affine variety U_{σ} as described above. For $\sigma_1, \sigma_2 \in \Sigma$ we have inclusions of $\tau = \sigma_1 \cap \sigma_2$ in σ_1 and σ_2. Dualising these inclusions we get maps $S_{\sigma_i} = \sigma_i^{\vee} \cap M \to \tau \cap M$. As described above, these induce maps $U_{\tau} \to U_{\sigma_i}$ and those maps are open inclusions. Checking compatibility this enables us to glue the varieties U_{σ} together and obtain a normal, separated variety X_{Σ}. The set $\tau = \{0\} \subset N$ gives $\tau^{\vee} \cap M = M$ and the corresponding inclusions $U_{\tau} = T_N = (\mathbb{C}^*)^n \hookrightarrow U_{\sigma}$, $\sigma \in \Sigma$, are compatible and show that X_{Σ} is a toric variety. By Corollary 3.1.8 of [CLS11] all normal separated toric varieties arise in this way.
For later purposes we need to extend the definition of X_{Σ} to generalized fans Σ. In this case let $\sigma_0 = \bigcap_{\sigma \in \Sigma} \sigma$. Then σ_0 is a subspace of $N_{\mathbb{R}}$. Define $\overline{N} = N/(\sigma_0 \cap N)$. When taking the quotient, we obtain that $\overline{\sigma} = \sigma/\sigma_0 \in \overline{N}_R = N_{\mathbb{R}}/\sigma_0$ is strongly convex for all cones $\sigma \in \Sigma$, hence $\overline{\Sigma} = \{\overline{\sigma}; \sigma \in \Sigma\}$ is an ordinary fan. Define $X_{\Sigma} = X_{\overline{\Sigma}}$.
We have seen above that maps of semigroup algebras $S \to S'$ induce (contravariantly) maps of their corresponding affine toric varieties $\operatorname{Spec}(\mathbb{C}[S']) \to \operatorname{Spec}(\mathbb{C}[S])$. One

checks that these maps are equivariant with respect to the torus actions. Algebraic morphisms of toric varieties which are torus-equivariant are called **toric morphisms**. Now assume we have cones $\sigma \subset N_\mathbb{R}$, $\sigma' \subset N'_\mathbb{R}$ and a linear map $\varphi : N \to N'$ such that the induced map $\varphi_\mathbb{R} : N_\mathbb{R} \to N'_\mathbb{R}$ satisfies $\varphi_\mathbb{R}(\sigma) \subset \sigma'$. Then the dual map $\varphi^*_\mathbb{R} : M'_\mathbb{R} \to M_\mathbb{R}$ sends $(\sigma')^\vee$ into $\sigma^\vee$ and we have a map $\varphi^* : (\sigma')^\vee \cap M' \to \sigma^\vee \cap M$ of semialgebras. Hence we see that a map of cones $\sigma \to \sigma'$ induces a toric morphism $U_\sigma \to U_{\sigma'}$ of affine toric varieties. This construction generalizes to fans and normal toric varieties. Let Σ be a fan in $N_\mathbb{R}$ and Σ' a fan in $N'_\mathbb{R}$ then a linear map $\varphi : N \to N'$ is called compatible with Σ, Σ' if for every cone $\sigma \in \Sigma$ there is a cone $\sigma' \in \Sigma'$ such that $\varphi_\mathbb{R}(\sigma) \subset \sigma'$. Such maps then induce maps of the toric varieties $X_\Sigma \to X_{\Sigma'}$ by glueing the corresponding maps $U_\sigma \to U_{\sigma'}$. If $N = N'$ and $\varphi = \mathrm{id}_N$ is compatible with Σ, Σ' then we say that Σ **refines** Σ'.

2.3.2. The orbit-cone correspondence and torus-invariant divisors

For $\sigma \subset N_\mathbb{R}$ a polyhedral cone we define the **distinguished point** $\gamma_\sigma \in U_\sigma$ as follows: as before let $S_\sigma = \sigma^\vee \cap M$, then we set $\mathfrak{m} = \mathrm{Span}(\chi^m; m \in S_\sigma \setminus \{0\})$. It is easy to see that $\mathfrak{m}$ is a maximal ideal and therefore corresponds to a unique closed point $\gamma_\sigma \in U_\sigma = \mathrm{Spec}(\mathbb{C}[S_\sigma])$. Now assume we have a fan Σ in $N_\mathbb{R}$ and a cone $\sigma \in \Sigma$. We define $O(\sigma) \subset X_\Sigma$ to be the orbit of γ_σ in X_Σ under the action of the torus T_n and $V(\sigma) = \overline{O(\sigma)}$ its closure. In this situation, $V(\sigma)$ is again a toric variety.
If $\Sigma \subset N_\mathbb{R}$ is a fan one can show that under this construction, the cones in Σ correspond bijectively to the orbits of the torus action on X_Σ. This bijection is called the **orbit cone correspondence**. Additionally we have $\dim(\sigma) + \dim(O(\sigma)) = n$ and so $\sigma = \{0\}$ corresponds to the torus T_N itself whereas full-dimensional cones $\sigma \in \Sigma(n)$ correspond to fixed points of the action.
Now let $\rho \in \Sigma(1)$ be one-dimensional (i.e. a ray). Then $V(\rho)$ is irreducible of codimension 1 and hence defines a **torus-invariant prime divisor** on X_Σ. One can show that every element of the class group $\mathrm{Cl}(X_\Sigma)$ has a representative, which is a $\mathbb{Z}$-linear combination of elements D_ρ for $\rho \in \Sigma(1)$. In fact, for $\rho \in \Sigma(1)$ let $u_\rho \in N$ be the minimal generator of $\rho \cap N$. Then if the vectors $u_\rho \in N_\mathbb{R}$ span the entire space $N_\mathbb{R}$ we have an exact sequence

$$0 \to M \to \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z} D_\rho \to \mathrm{Cl}(X_\Sigma) \to 0,$$

where $m \in M$ maps to $\mathrm{div}(\chi^m) = \sum_{\rho \in \Sigma(1)} \langle m, u_\sigma \rangle D_\rho$ and a $\mathbb{Z}$-linear combination of divisors D_ρ maps to its equivalence class in $\mathrm{Cl}(X_\Sigma)$. One proves that the condition $\mathrm{Span}(u_\rho; \rho \in \Sigma(1)) = N_\mathbb{R}$ is equivalent to the variety X_Σ having no **torus factor**, i.e. it is not a product of a nontrivial torus with another variety.

2.3.3. Examples and constructions

Before moving on with the general theory, let us give some examples for fans and their corresponding toric varieties. We will use them later to identify some of the varieties we encounter.

Affine and projective space

We have seen above that we obtain the torus $T_n = (\mathbb{C}^*)^n$ as the toric variety of the fan $\{0\} \subset \mathbb{Z}^n$. The next simplest example of a toric variety of dimension n is the affine space $\mathbb{C}^n$ itself. It contains the torus T_n and the componentwise multiplication extends the action of T_n on itself. As $\mathbb{C}^n$ is indeed affine, it is of the form U_σ for some cone $\sigma \subset \mathbb{Z}^r$ and modulo isomorphism we have

$$\sigma = \mathrm{Cone}(e_1, \ldots, e_n) \subset (\mathbb{Z}^n)_{\mathbb{R}}.$$

Here we choose $N = \mathbb{Z}^r$ and thus we can identify $M = \mathbb{Z}^r$ via the standard bilinear form on $\mathbb{Z}^r \times \mathbb{Z}^r$. As above we denote $e_1, \ldots, e_n \in N$ the standard basis of $\mathbb{Z}^r$ and let $e_1^*, \ldots, e_n^* \in M$ be the dual basis. One easily sees that

$$S_\sigma = \sigma^\vee \cap M = \mathrm{Cone}(e_1^*, \ldots, e_n^*) \cap M = \sum_{i=1}^{n} \mathbb{N}e_i^* = \mathbb{N}^r.$$

Then one obtains $\mathbb{C}(S_\sigma) = \mathbb{C}[x_1, \ldots, x_n]$, where $\chi^m \in S_\sigma$ is identified with $x^m = x_1^{m_1} x_2^{m_2} \cdot \ldots \cdot x_n^{m_n}$. Thus we find $U_\sigma = \mathrm{Spec}(\mathbb{C}[x_1, \ldots, x_n]) = \mathbb{C}^n$. Here the torus fixed point γ_σ corresponds exactly to the origin $0 \in \mathbb{C}^n$. For $n = 2$ a picture of σ is given in Figure 2.2.

To give a more elaborated example, consider the projective space $\mathbb{P}^n$. This is indeed a toric variety. To see this note that $\mathbb{P}^n$ contains the torus

$$(\mathbb{C}^*)^n \hookrightarrow \mathbb{P}^n, (t_1, \ldots, t_n) \mapsto [t_1 : t_2 : \ldots : t_n : (t_1 t_2 \ldots t_n)^{-1}]$$

whose action on itself (by componentwise multiplication) extends continuously to $\mathbb{P}^n$. To obtain its corresponding fan $\Sigma_{\mathbb{P}^n}$ let $e_1, \ldots, e_n$ be the standard basis of $N = \mathbb{Z}^n$ and $e_0 = -e_1 - e_2 \ldots - e_n$. Then $\Sigma_{\mathbb{P}^n}$ consists of all cones spanned by proper subsets of $\{e_0, e_1, \ldots, e_n\}$. For $n = 2$ this fan is pictured in Figure 2.2.

To see that this fan gives the variety $\mathbb{P}^n$ one can observe that the $n+1$ maximal cones $\sigma_i = \mathrm{Cone}(e_0, e_1, \ldots, \widehat{e_i}, \ldots, e_n)$ of $\Sigma_{\mathbb{P}^n}$ are all isomorphic to $\mathrm{Cone}(e_1, \ldots, e_n)$. They correspond to the $n+1$ affine open subsets

$$U_i = \{[x_0 : \ldots : x_n] \in \mathbb{P}^n; x_i \neq 0\} \subset \mathbb{P}^n.$$

Then one can check that these pieces glue in the expected way when following the construction in Section 2.3.1 (see for instance Example 3.1.10 in [CLS11]). We will continue to develop more theory and prove that $\Sigma_{\mathbb{P}^n}$ corresponds to $\mathbb{P}^n$ in Section 5.1.1 using the standard quotient description of $\mathbb{P}^n$.

We remark that via the orbit-cone correspondence, the cones σ_i correspond to the $n+1$ coordinate points in $\mathbb{P}^n$, which are fixed under the action of $(\mathbb{C}^*)^n \subset \mathbb{P}^n$. The rays $\rho_i = \mathrm{Cone}(e_i)$ correspond to the orbit $O(\rho_i) \subset \mathbb{P}^n$ of all points $[x] \in \mathbb{P}^n$ with $x_i = 0$ and $x_j \neq 0$ for $j \neq i$. The closure $V(\rho_i)$ of this orbit is exactly the torus-invariant divisor

D_{ρ_i} that is the vanishing locus of the homogenious polynomial x_i.

Blowup of points

We now come to an essential operation for toric varieties: the blow-up of a torus fixed point. It turns out that this corresponds to a nice operation on the fan defining our toric variety. The following description of blow-up and star-subdivision is essentially taken from Definition 3.3.13 and Proposition 3.3.15 of [CLS11].

Let X_Σ be a toric variety, where Σ is a fan in $N_\mathbb{R}$, and let $\gamma_\sigma \in X_\Sigma$ be a fixed point of the torus action. Here $\sigma \in \Sigma(n)$ is the full-dimensional cone associated to γ_σ via the orbit-cone correspondence. Assume that σ is smooth, i.e. the minimal generators $u_1, \dots, u_n$ of its rays form a $\mathbb{Z}$-basis of N. Then the blow-up $\mathrm{Bl}_{\gamma_\sigma} X_\Sigma$ of X_Σ at γ_σ is again a toric variety. The corresponding fan is the **star-subdivision $\Sigma^*(\sigma)$ of Σ along σ**. It is constructed as follows: let $u_0 = u_1 + \dots + u_n$ be the sum of all minmal ray-generators of σ and let $\Sigma'(\sigma)$ be the set of cones generated by subsets of $\{u_0, u_1, \dots, u_n\}$ not containing $\{u_1, \dots, u_n\}$. Then

$$\Sigma^*(\sigma) = (\Sigma \setminus \{\sigma\}) \cup \Sigma'(\sigma).$$

This fan is a refinement of Σ and the induced toric morphism $X_{\Sigma^*(\sigma)} \to X_\Sigma$ makes $X_{\Sigma^*(\sigma)}$ the blow-up of X_Σ at γ_σ (see Proposition 3.3.15, [CLS11]). We also note that all the newly introduced cones are simplicial, so in particular the blow-up of a simplicial fan remains simplicial. For an example see Figure 2.2.

Products and fibre bundles

Next we will construct products of toric varieties and their fans. It turns out that this is only a special case of the more general situation of a toric fibration, i.e. a fibre bundle over a toric variety that has a natural structure of a toric variety itself.

First let Σ_1, Σ_2 be fans in $(N_1)_\mathbb{R}, (N_2)_\mathbb{R}$, respectively and consider their toric varieties X_{Σ_1} and X_{Σ_2}. When $T^1 \subset X_{\Sigma_1}$ and $T^2 \subset X_{\Sigma_2}$ are the corresponding tori, the product $T^1 \times T^2 \subset X_{\Sigma_1} \times X_{\Sigma_2}$ is also a torus and one sees easily, that this makes $X_{\Sigma_1} \times X_{\Sigma_2}$ a toric variety itself. By Proposition 3.1.14 in [CLS11] the corresponding fan is given by

$$\Sigma_1 \times \Sigma_2 = \{\sigma_1 \times \sigma_2 \subset (N_1)_\mathbb{R} \times (N_2)_\mathbb{R} = (N_1 \oplus N_2)_\mathbb{R}; \sigma_1 \in \Sigma_1, \sigma_2 \in \Sigma_2\}.$$

Now we turn to a more general situation. Assume we have a surjective morphism $\overline{\varphi} : N \to N'$ of finitely generated lattices and let $N_0 \subset N$ be its kernel. Assume further that we have fans Σ, Σ' in $N_\mathbb{R}, N'_\mathbb{R}$ compatible with $\overline{\varphi}$ inducing a toric morphism $\varphi : X_\Sigma \to X'_\Sigma$. Define $\Sigma_0 = \{\sigma \in \Sigma; \sigma \subset (N_0)_\mathbb{R}\}$, then this is a subfan of Σ.

Following Definition 3.3.18. in [CLS11] we say that Σ is **split by Σ'** and Σ_0 if there exists a subfan $\widehat{\Sigma} \subset \Sigma$ such that

- $\overline{\varphi}_\mathbb{R}$ maps the cones $\widehat{\sigma} \in \widehat{\Sigma}$ bijectively to cones $\sigma' \in \Sigma'$ such that $\overline{\varphi}(\widehat{\sigma} \cap N) = \sigma' \cap N'$. Furthermore this correspondence $\widehat{\sigma} \to \sigma'$ is a bijection $\widehat{\Sigma} \to \Sigma'$.

- For cones $\widehat{\sigma} \in \widehat{\Sigma}$ and $\sigma_0 \in \Sigma_0$, their sum $\widehat{\sigma} + \sigma_0$ lies in Σ and all cones in Σ are of this form.

Then we cite the following result.

Theorem (see Theorem 3.3.19. in [CLS11])**.** If Σ is split by Σ' and Σ_0 as above then X_Σ is a locally trivial fibre bundle over $X_{\Sigma'}$ with fibre X_{Σ_0, N_0}, i.e., $X_{\Sigma'}$ has a cover by affine open subsets U satisfying

$$\varphi^{-1}(U) \cong X_{\Sigma_0, N_0} \times U.$$

In particular, all fibres of $X_\Sigma \to X_{\Sigma'}$ are isomorphic to X_{Σ_0, N_0}.

Fibre bundles as above will occur naturally in Section 5. We want to give one example to illustrate the construction. In the previous paragraph we already considered the fan of the variety $\mathrm{Bl}_1\mathbb{P}^2$. Its fan, given in Figure 2.2, is only one of three natural fans describing this variety. The three possibilities correspond to the three coordinate points in $\mathbb{P}^2$, which we can blow up.
For convenience consider now the blow-up of the coordinate point $[0:0:1]$, corresponding to the cone $\sigma_2 = \mathrm{Cone}(e_0, e_1)$. Its fan $\Sigma \subset N_\mathbb{R} = (\mathbb{Z}^2)_\mathbb{R}$ is depicted in Figure 2.1.
Let $N' = \mathbb{Z}$ and let $\phi : N \to N'$ be the (surjective) projection on the first coordinate.

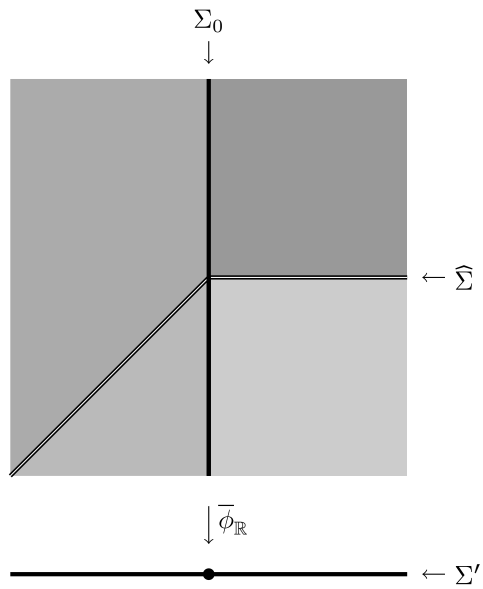
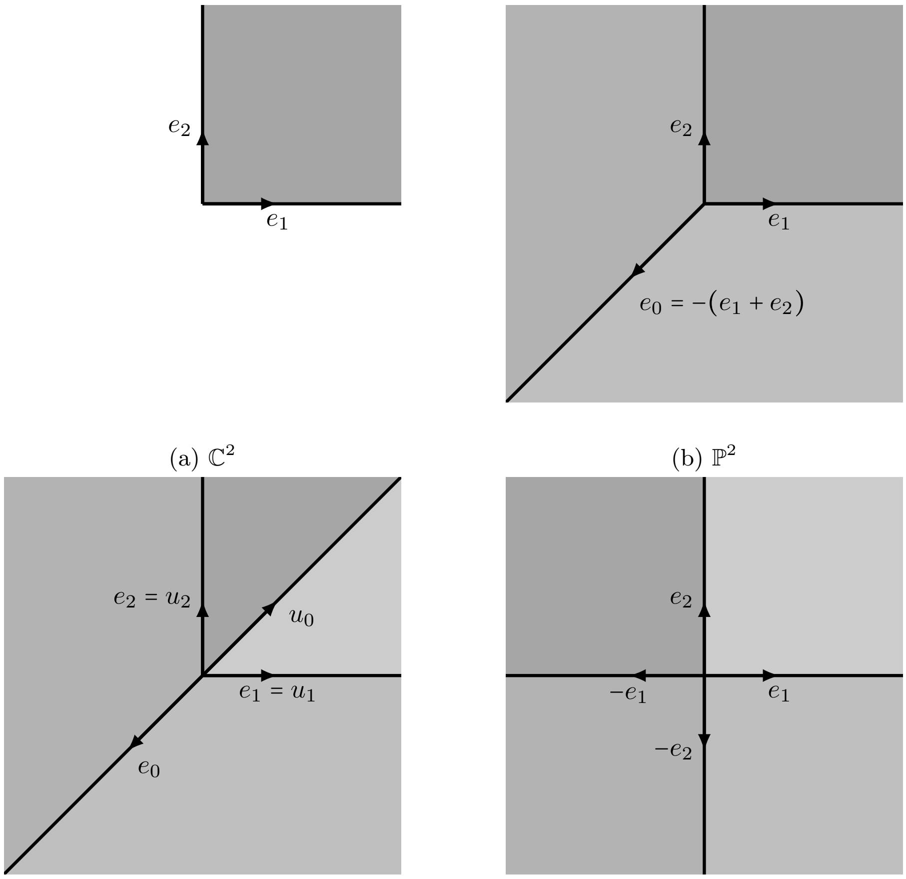


Figure 2.1.: $\mathrm{Bl}_1\mathbb{P}^2$ as a toric fibration

Then taking $\Sigma' = \{\mathbb{R}_{\le 0}, \{0\}, \mathbb{R}_{\ge 0}\} \subset N'_\mathbb{R}$ we see that the map ϕ is compatible with the fans Σ, Σ'. Its kernel is $N_0 = \mathbb{Z}e_2 \subset N$ and correspondingly $\Sigma_0 = \{\mathbb{R}_{\le 0}e_2, \{0\}, \mathbb{R}_{\ge 0}e_2\} \subset \Sigma$.

Choosing $\widehat{\Sigma} = \{\mathbb{R}_{\geq 0}e_0, \{0\}, \mathbb{R}_{\geq 0}e_1\} \subset \Sigma$ one verifies that Σ is split by Σ' and Σ_0. Now $\Sigma' \cong \Sigma_0 \cong \Sigma_{\mathbb{P}^1}$ and thus $X_\Sigma = \mathrm{Bl}_{[0:0:1]}\mathbb{P}^2$ is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ via the continuous extension of the map $[x_0 : x_1 : x_2] \mapsto [x_0 : x_1]$ defined on $\mathbb{P}^2 \backslash \{[0:0:1]\}$.



(a) $\mathbb{C}^2$ (b) $\mathbb{P}^2$

(c) $\mathrm{Bl}_1\mathbb{P}^2$ (d) $\mathbb{P}^1 \times \mathbb{P}^1$

Figure 2.2.: Examples of fans

Projective bundles

When we have a fibre bundle as above, we may wish to have a more concrete way to identify it. An important class of toric fibre bundles on a toric variety $X_{\Sigma'}$ arise from vector bundles on $X_{\Sigma'}$. We will give a general construction below. This paragraph is based on Chapter 7.0 in [CLS11].

For this let X be an algebraic variety and let $\pi : V \to X$ be a vector bundle of rank

n on X. By definition we have trivializations $h_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^n$ for open sets U_i covering X. For two such trivializations, we may look at the transition map

$$h_i \circ h_j^{-1} : (U_i \cap U_j) \times \mathbb{C}^n \to (U_i \cap U_j) \times \mathbb{C}^n, (u, v) \mapsto (u, g_{ij}(u)v),$$

where $g_{ij} : U_i \cap U_j \to \mathrm{GL}(n, \mathbb{C})$ are transition functions. Remember that one can obtain the total space V of the vector bundle by glueing the pieces $U_i \times \mathbb{C}^n$ using this transition data. Now note that any element A of $\mathrm{GL}(n, \mathbb{C})$ also gives rise to a linear automorphism $\bar{A}$ of the projective space $\mathbb{P}^{n-1}$. We define the **projective bundle** $\mathbb{P}(V)$ to be the variety obtained by glueing the pieces $U_i \times \mathbb{P}^{n-1}$ along the isomorphisms

$$\mathrm{id} \times \bar{g}_{ij} : U_i \cap U_j \times \mathbb{P}^{n-1} \to U_i \cap U_j \times \mathbb{P}^{n-1},$$

together with the induced map $\mathbb{P}(V) \to X$. In [CLS11] it is shown that this morphism is projective, making $\mathbb{P}(V)$ a $\mathbb{P}^{n-1}$-bundle over X.
Given a locally free sheaf $\mathcal{E}$ of rank n on X this corresponds to a vector bundle $V_{\mathcal{E}} \to X$ whose sheaf of sections is $\mathcal{E}$. Then we define $\mathbb{P}(\mathcal{E})$ to be the projective bundle of the dual bundle $V_{\mathcal{E}}^{\vee}$.
In Proposition 7.3.3 of [CLS11] the authors give a description of the fan $\Sigma_{\mathcal{E}}$ with toric variety $\mathbb{P}(\mathcal{E})$ for a vector bundle $\mathcal{E}$ of rank $n+1$ on a toric variety X_Σ, which is of the form

$$\mathcal{E} = \mathcal{O}_{X_\Sigma}(D_0) \oplus \ldots \mathcal{O}_{X_\Sigma}(D_n).$$

Here $D_0, \ldots, D_n$ are Cartier divisors on the toric variety X_Σ and $\mathcal{O}_{X_\Sigma} \to X_\Sigma$ the corresponding line bundles.
We do not present the construction of the fan $\Sigma_{\mathcal{E}}$ in greater detail here. Varieties of this form will occur in our main series of examples and we will identify the corresponding divisors $D_0, \ldots, D_n$.

2.3.4. Toric varieties as quotients

Returning to the general theory we want to represent a toric variety as a quotient. We will give rigorous definitions in what sense we mean quotient here in Chapter 3. Remember that when computing the Class group of a toric variety X_Σ without torus factor, we had an exact sequence

$$0 \to M \to \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z}D_\rho \to \mathrm{Cl}(X_\Sigma) \to 0. \tag{2.1}$$

By applying $\mathrm{Hom}(-, \mathbb{C}^*)$ and using that $\mathbb{C}^*$ is divisible we obtain

$$1 \to \mathrm{Hom}(\mathrm{Cl}(X_\Sigma), \mathbb{C}^*) \to (\mathbb{C}^*)^{\Sigma(1)} \to T_N \to 1, \tag{2.2}$$

where we use the identification $T_N \cong \mathrm{Hom}(M, \mathbb{C}^*)$. With above sequence, the group $G = \mathrm{Hom}(\mathrm{Cl}(X_\Sigma), \mathbb{C}^*)$ can be regarded as a (closed) subgroup of $(\mathbb{C}^*)^{\Sigma(1)}$ and the quotient $(\mathbb{C}^*)^{\Sigma(1)}/G$ is isomorphic to the torus. The hope is that when we enlarge $(\mathbb{C}^*)^{\Sigma(1)}$ enough inside $\mathbb{C}^{\Sigma(1)}$ and quotient by G, the resulting quotient space will be

the entire toric variety X_Σ instead of the dense open subset T_N.
In order to find a suitable subset of $\mathbb{C}^{\Sigma(1)}$ to quotient by G, we define

$$S = \mathbb{C}[x_\rho; \rho \in \Sigma(1)],$$

which is the coordinate ring of $\mathbb{C}^{\Sigma(1)}$. It is also called the **total coordinate ring**. We define in this ring the **irrelevant ideal**

$$B(\Sigma) = (x^{\hat{\sigma}}; \sigma \in \Sigma) \subset S,$$

where $x^{\hat{\sigma}} = \prod_{\rho \notin \sigma(1)} x_\rho$. Then its vanishing set $Z(\Sigma) = V(B(\Sigma)) \subset \mathbb{C}^{\Sigma(1)}$ is exactly the set that we have to remove from $\mathbb{C}^{\Sigma(1)}$ before taking the quotient.
To see this we note that $\mathbb{C}^{\Sigma(1)}$ and $\mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma)$ are also toric varieties. If $\{e_\sigma; \sigma \in \Sigma(1)\}$ is the standard basis of $\mathbb{Z}^{\Sigma(1)}$, then the cone $\mathrm{Cone}(e_\rho; \rho \in \Sigma(1))$ and all its faces form a fan $\tilde{\Sigma}_0$ in $\mathbb{R}^{\Sigma(1)}$, whose corresponding toric variety is $\mathbb{C}^{\Sigma(1)}$ itself.
Now for all $\sigma \in \Sigma$ define $\tilde{\sigma} = \mathrm{Cone}(e_\rho; \rho \in \sigma(1)) \subset \mathbb{R}^{\Sigma(1)}$. Then the fan $\tilde{\Sigma} = \{\tau; \tau \preceq \tilde{\sigma}, \sigma \in \Sigma\}$ is a subfan of $\tilde{\Sigma}_0$. By the orbit-cone-correspondence, one sees that the inclusion $\tilde{\Sigma} \subset \tilde{\Sigma}_0$ induces the inclusion $\mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma) \subset \mathbb{C}^{\Sigma(1)}$.
Now there is also a canonical map $\mathbb{R}^{\Sigma(1)} \to N_\mathbb{R}, e_\rho \mapsto u_\rho$ which is compatible with $\tilde{\Sigma}$ and Σ. Hence it induces a map $\tilde{\pi} : \mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma) \to X_\Sigma$. This map $\tilde{\pi}$ can be seen to be an extension of the map $(\mathbb{C}^*)^{\Sigma(1)} \to T_N$ from the sequence in (2.2). Finally it is shown in Theorem 5.1.11 of [CLS11] that $\tilde{\pi}|_{\mathbb{C}^{\Sigma(1)} \setminus Z(\sigma)}$ is an almost geometric quotient (for a definition see Chapter 3).
In Chapter 3 we are going to look at a more general situation, where a closed subgroup of a torus acts on an affine space. The above situation is an important example of such an action, which we will keep in mind.

2.4. Gale duality

The concept of **Gale duality** from linear algebra will be important in a number of constructions in this thesis. Our account here is based on Chapter 14.3 in [CLS11].
The setting is the following: we have a real, finite-dimensional vector space W and elements $\beta_1, \ldots, \beta_r \in W$ which span the entire space. Note that we allow them to be linearly dependent or even some of them to be zero. This induces a surjective map $\gamma : \mathbb{R}^r \to W$ by $e_i \mapsto \beta_i$. Let V denote the kernel of this map and $\delta : V \to \mathbb{R}^r$ the inclusion in $\mathbb{R}^r$. Then we obtain an exact sequence

$$0 \to V \xrightarrow{\delta} \mathbb{R}^r \xrightarrow{\gamma} W \to 0. \tag{2.3}$$

Taking the dual of this sequence gives us another exact sequence as follows

$$0 \to W^* \xrightarrow{\gamma^*} \mathbb{R}^r \xrightarrow{\delta^*} V^* \to 0, \tag{2.4}$$

where we identify $(\mathbb{R}^r)^*$ with $\mathbb{R}^r$ using the standard Euclidean inner product. We denote $\nu_i = \delta^*(e_i) \in V^*$ for $i = 1, \dots r$. Note that by definition we have $\delta(m) = (\langle m, \nu_1 \rangle, \dots, \langle m, \nu_r \rangle)$.
This construction gives us two r-tuples of vectors: $\boldsymbol{\beta} = (\beta_1, \dots, \beta_r) \in W^r$ and $\boldsymbol{\nu} = (\nu_1, \dots, \nu_r) \in (V^*)^r$, which are said to be Gale dual to one another. Usually, by abuse of notation, we will think of $\boldsymbol{\beta}$ and $\boldsymbol{\nu}$ as sets (rather than ordered tuples) of vectors in W and V^*, respectively.
We will need the following result in a later section and for lack of a reference, we prove it here.

Lemma 2.4. Let $\boldsymbol{\nu}, \boldsymbol{\beta}$ be Gale dual vector configurations. Then

$$\nu_i \in \operatorname{Cone}(\nu_1, \dots, \widehat{\nu_i}, \dots, \nu_r) \iff \beta_i \notin \operatorname{Cone}(\beta_1, \dots, \widehat{\beta_i}, \dots, \beta_r).$$

By duality, the equivalent statement with the roles of $\boldsymbol{\nu}$ and $\boldsymbol{\beta}$ swapped holds as well.

Proof. We are going to prove the dual statement by a chain of equivalences.

$$\begin{aligned} & \beta_i \in \operatorname{Cone}(\beta_1, \dots, \widehat{\beta_i}, \dots, \beta_r) \\ \iff & \text{There exist } a_j \geq 0 \text{ such that } \sum_{j \neq i} a_j \beta_j - \beta_i = 0. \\ \iff & \text{There exist } a_j \geq 0 \text{ such that } (a_1, \dots, -1, \dots, a_r) \in \ker(\gamma_{\mathbb{R}}). \\ \iff & \text{There exist } v \in V \text{ such that } \langle v, \nu_j \rangle \begin{cases} \geq 0, & j \neq i \\ < 0, & j = i \end{cases}. \\ \iff & \nu_i \notin \operatorname{Cone}(\nu_1, \dots, \widehat{\nu_i}, \dots, \nu_r). \end{aligned}$$

Here the last equivalence follows from the fact that every convex polyhedral cone may be separated from a vector not contained in it by a linear inequality. ☐

2.5. Vector configurations and oriented matroids

The following short introduction to vector configurations and other related topics is based on Sections 6.2-6.4 and 7.1-7.4 from [Zie95]. These notions will play a role when dealing with wall-crossings in later sections.

2.5.1. Signs and sign vectors

Oriented matroids are combinatorial objects that have a wide range of applications. For our purposes we will look at a specific situation, namely vector configurations and hyperplane arrangements, in which they naturally show up. We denote by $\{-, 0, +\}$ the set of **signs** and elements of $\{-, 0, +\}^m$ are called **sign vectors**. We endow the set of signs with an order, such that $- > 0, + > 0$ and $-$ and $+$ are incomparable. This induces an order on the set of sign vectors by defining $a = (a_i)_{i=1}^m \geq b = (b_i)_{i=1}^m$ if $a_i \geq b_i$

for all $i = 1, \dots, m$. For instance we have

$$(0, +, 0, -) \le (-, +, 0, -)$$

whereas

$$(+, +, 0, -) \text{ and } (0, 0, -, -)$$

are not comparable as in the second component $+ > 0$ but in the third $0 < -$. On the other hand

$$(+, +, 0, -) \text{ and } (+, +, 0, +)$$

are incomparable because in the last component $-$ and $+$ are not comparable.
For any set $S \subset \{-, 0, +\}^m$ of sign vectors we denote by

$$\mathrm{MIN}(S) = \{u \in S \setminus \{0\};\ \text{there is no } u' < u \text{ with } u' \in S \setminus \{0\}\}$$

the set of **minimal nonzero sign vectors** in S.
An **oriented matroid** can be specified as a set of sign vectors fulfilling certain axioms (see [Zie95], Definition 7.21). However, we will not have to deal with the axioms directly as all oriented matroids that will arise in our studies will be induced from other data, ensuring that the axioms are fulfilled. For $m \ge 1$ an arbitrary positive integer we denote

$$\mathrm{SIGN} : \mathbb{R}^m \to \{-, 0, +\}^m, (a_1, \dots, a_m) \mapsto (\mathrm{sign}(a_1), \dots, \mathrm{sign}(a_m))$$

the sign-function applied componentwise. When we have an entire subspace $U \subset \mathbb{R}^m$ then $\mathrm{SIGN}(U) \subset \{-, 0, +\}^m$ will be a set of sign vectors. We will obtain our oriented matroids by applying SIGN to subspaces U.

2.5.2. Vector configurations

A tuple $v_1, \dots, v_r \in \mathbb{R}^n$ of r vectors in $\mathbb{R}^n$ is called a **vector configuration**. Note that all constructions will still work if the vectors are contained in an arbitrary finite-dimensional real vector space (i.e. we do not need to choose a basis). However for simplicity we will work with the standard n-dimensional space $\mathbb{R}^n$ as ambient vector space. In the following we will assume that the vectors span the entire space. We denote $V \in \mathbb{R}^{n \times r}$ the matrix obtained by concatenating the column vectors $v_1, \dots, v_r$. Then

$$\mathrm{Dep}(V) = \{a \in \mathbb{R}^r; Va = 0\}$$

is the set of **linear dependencies** of the vector configuration. However, we will not be interested in this entire space but only at the signs of the components of vectors in $\mathrm{Dep}(V)$. We define the **(signed) vectors** of V as

$$\mathcal{V}(V) = \mathrm{SIGN}(\mathrm{Dep}(V)) \subset \{-, 0, +\}^r.$$

Those sign vectors in $\mathcal{V}(V)$ which are nonzero and minimal with respect to the ordering described above are called the **(signed) circuits** of V and denoted by

$$\mathcal{C}(V) = \mathrm{MIN}(\mathrm{Dep}(V)).$$

As a dual construction to this, we define the **value vectors** of V as

$$\mathrm{Val}(V) = \{V'c = c \circ V; c \in (\mathbb{R}^n)^*\} \subset (\mathbb{R}^r)^*.$$

This can be seen as the space of all row vectors obtained by applying linear functionals to the vectors in V. Identifying $(\mathbb{R}^r)^*$ with $\mathbb{R}^r$ via the standard Euclidean inner product, we may apply the SIGN-operation on $(\mathbb{R}^r)^*$ and we define the **(signed) covectors** of V as

$$\mathcal{V}^*(V) = \mathrm{SIGN}(\mathrm{Val}(V)) \subset \{-, 0, +\}^r.$$

Similar to above, the minimal nonzero vectors in $\mathcal{V}^*(V)$ are called the **(signed) cocircuits** of the vector configuration V and are denoted by

$$\mathcal{C}^*(V) = \mathrm{MIN}(\mathrm{Val}(V)).$$

They correspond to (oriented) hyperplanes H in $\mathbb{R}^n$ such that the vectors of V contained in H span the hyperplane H.
The oriented matroid $\mathcal{M}(V)$ is a combinatorial structure given by the sets $\mathcal{V}(V), \mathcal{C}(V)$, $\mathcal{V}^*(V)$ and $\mathcal{C}^*(V)$. An oriented matroid obtained in such a way from a vector configuration is called a **realizable oriented matroid**. One can show that any of these four sets determines the others (see [Zie95], Corollary 6.9).

2.5.3. Operations on realizable oriented matroids

Duality : For a realizable oriented matroid $\mathcal{M}$ as above, we can define the **dual oriented matroid** $\mathcal{M}^*$ by taking as vectors of $\mathcal{M}^*$ the covectors of $\mathcal{M}$ and vice versa and applying the same switch to circuits and cocircuits. Then it turns out (see [Zie95], Theorem 6.14 and Corollary 6.15) that $\mathcal{M}^*$ is realizable and comes from the vector configuration $w_1, \dots, w_r \in (\mathbb{R}^{r-n})^*$ which is Gale dual to the configuration $v_1, \dots, v_r \in \mathbb{R}^n$.
We repeat again how to obtain these vectors. The matrix $V \in \mathbb{R}^{n\times r}$ formed by the column vectors $v_1, \dots, v_r$ defines a linear map $\mathbb{R}^r \to \mathbb{R}^n$, which is surjective. Its kernel has dimension $r - n$ and we can choose a matrix $G \in \mathbb{R}^{r\times(r-n)}$ such that the image of the corresponding map $\mathbb{R}^{r-n} \to \mathbb{R}^r$ is the kernel of V. Then the vectors $w_1, \dots, w_r \in (\mathbb{R}^r)^* \cong \mathbb{R}^{1\times r}$ are exactly the rows of G. The matrix G is unique up to column operations and one checks that this does not affect the vectors, covectors, etc. of the vector configuration $w_1, \dots, w_n$.
Deletion and Contraction : One important operation on a vector configuration is to simply leave out, or **delete**, the i-th vector v_i. We denote by $V \setminus v_i \in \mathbb{R}^{n\times(r-1)}$ the matrix obtained from V by erasing the i-th column and for a sign vector $\mathrm{U} \in \{-, 0, +\}^r$ we denote $\mathrm{U} \setminus i \in \{-, 0, +\}^{r-1}$ the vector obtained by deleting the i-th entry. Then we

obtain a new oriented matroid $\mathcal{M}(V \smallsetminus v_i)$ and by [Zie95], Proposition 6.11, we have

$$\mathcal{V}(V \smallsetminus v_i) = \{\mathrm{U} \smallsetminus i; \mathrm{U} \in \mathcal{V}(V), \mathrm{U}_i = 0\} \quad \mathcal{V}^*(V \smallsetminus v_i) = \{\mathrm{U} \smallsetminus i; \mathrm{U} \in \mathcal{V}^*(V)\}$$
$$\mathcal{C}(V \smallsetminus v_i) = \{\mathrm{C} \smallsetminus i; \mathrm{C} \in \mathcal{C}(V), \mathrm{C}_i = 0\} \quad \mathcal{C}^*(V \smallsetminus v_i) = \mathrm{MIN}\{\mathrm{C} \smallsetminus i; \mathrm{C} \in \mathcal{C}^*(V)\}.$$

The dual operation to deletion of a vector v_i is the **contraction** of the corresponding vector w_i in the Gale dual vector configuration G. This means the new vector configuration G/w_i is given by

$$[w_1], [w_2], \ldots, [w_{i-1}], [w_{i+1}], \ldots, [w_r] \in (\mathbb{R}^{r-n})^*/\mathbb{R}w_i.$$

One checks that the oriented matroid $\mathcal{M}(G/w_i)$ is exactly the dual of the matroid $\mathcal{M}(V \smallsetminus v_i)$.

3. Toric quotients and the secondary fan

In this chapter we will look at the action of a closed subgroup $G \subset (\mathbb{C}^*)^r$ on the affine space $\mathbb{C}^r$ by componentwise multiplication. We will see that characters χ of the group G give us a possibility to define a quotient variety for this action and we will start investigating how this quotient depends on χ and how to compute it. This will lead to the definition of the secondary fan, which partitions the space of characters into pieces, where the quotient is essentially the same. The following chapter is based on Chapter 14 in [CLS11]. In Section 3.2.2 we expand an alternative approach that was only mentioned in [CLS11] and in Section 3.4.3 we give a more explicit construction of the secondary fan. However, we do not present any original results in the following chapter.

3.1. Geometric Invariant Theory

3.1.1. Group actions and quotients

Let G be a group which carries the structure of an algebraic variety. Assume that inversion and multiplication are algebraic maps $G \to G$ and $G \times G \to G$, respectively. In this case G is called an **algebraic group**. Given two algebraic groups G and G' a map $\varphi : G \to G'$ is called a **morphism of algebraic groups** if it is a group homomorphism that is also an algebraic map.
Now assume that G acts (set-theoretically) on a variety X via a map $\Psi : G \times X \to X$. If Ψ is algebraic too, we say that it is an **algebraic group action** and we write $G \curvearrowright X$. In the following we use the notation gx to denote $\Psi(g, x)$.
Assume that G acts on two spaces X and Y then a map $\varphi : X \to Y$ is called **equivariant** if $\varphi(gx) = g\varphi(x)$ for all $g \in G, x \in X$. On the other hand a map $\varphi : X \to Y$ with a G-action on X satisfying $\varphi(gx) = \varphi(x)$ for all $g \in G, x \in X$ is called an **invariant** map.
Now we can define what we mean by quotients of a G-action. Let G act on a variety X and let $\pi : X \to Y$ be a morphism. Then π is called a **categorical quotient** (of X by G) if it is invariant and for any variety Z and an invariant morphism $\psi : X \to Z$ there is a unique morphism $\chi : Y \to Z$ such that $\psi = \chi \circ \pi$.
The map π is a **good categorical quotient** if in addition it satisfies

- π is surjective,

- for $U \subset Y$ open the map $\pi^* : \mathcal{O}_Y(U) \to \mathcal{O}_X(\pi^{-1}(U))$ induces an isomorphism onto the set of invariant regular functions $(\pi^{-1}(U))^G$ on $\pi^{-1}(U)$,

- for $W \subset X$ closed and invariant its image $\pi(W)$ is closed,
- for $W_1, W_2 \subset X$ disjoint closed invariant subsets, their images $\pi(W_1), \pi(W_2)$ are disjoint.

In this case we write $\pi : X \to X/\!\!/G = Y$ for a good categorical quotient. It is unique up to isomorphism. If in addition π induces a bijection from the G-orbits in X to the points of $X/\!\!/G$ we say that it is a **geometric quotient**. If there is some dense open subset $U \subset Y$ such that $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ is a geometric quotient, we say that π is an **almost geometric quotient**.
Such algebraic actions and quotients naturally arise in various contexts in algebraic geometry, especially when trying to classify objects of a certain type "up to isomorphism". A classical problem is to study degree d hypersurfaces in $\mathbb{P}^n$ (identified with their defining polynomial in $\mathbb{C}[x_0, \dots, x_n]_d/\mathbb{C}^* = \mathbb{P}(\mathbb{C}[x_0, \dots, x_n]_d)$) modulo the natural action of $\mathrm{SL}(n+1) \curvearrowright \mathbb{C}[x_0, \dots, x_n]_d$. The area of mathematics dealing with such actions and defining quotient varieties is called **Geometric Invariant Theory**, or GIT.
David Mumford's monograph [Mum65] of the same name is seen by many as the starting point of the systematic study of such quotients. We will not try to give a history overview or a general introduction to GIT. For this we refer the reader to the excellent books of Mukai [MO03] and Newstead [New78], where the basic concepts of GIT are introduced on about the same level of abstraction as used in this thesis. The account above is largely based on [New78] and also on Chapter 5.0 of [CLS11].

3.1.2. Toric GIT

We will now restrict ourselves to a very special type of algebraic group action. Let $G \subset (\mathbb{C}^*)^r$ be a closed subgroup of the r-dimensional torus then it acts in a canonical way on the affine space $\mathbb{C}^r$. We mention that all subgroups of a torus which are irreducible subvarieties are also tori. Hence we have $G \cong (\mathbb{C}^*)^s \times H$ for H a finite abelian group. We want to look at quotients of $\mathbb{C}^r$ by G. For this purpose we lift the action of G on $\mathbb{C}^r$ to an action on the trivial line bundle $\mathcal{L} = \mathbb{C}^r \times \mathbb{C} \to \mathbb{C}^r$ over $\mathbb{C}^r$. We do so by using characters of the group G. We define the character group of G to be

$$\widehat{G} = \{\chi : G \to \mathbb{C}^*; \chi \text{ is a morphism of algebraic groups}\}.$$

Then every $\chi \in \widehat{G}$ defines an action of G on $\mathcal{L}$ by

$$g \cdot (x, t) = (gx, \chi(g)t).$$

The line bundle $\mathcal{L}$ with the above G-action is denoted by $\mathcal{L}_\chi$. Note that $(\mathcal{L}_\chi)^{\otimes d} \cong \mathcal{L}_{\chi^d}$. The action of G on $\mathcal{L}_\chi$ induces in a canonical way an action on the global sections of

$\mathcal{L}_\chi$. Let $s \in H^0(\mathbb{C}^r, \mathcal{L}_\chi)$ be given by $s(x) = (x, F(x))$ for $F \in \mathbb{C}[x_1, \ldots, x_r]$. Then we have

$$(gs)(x) = gs(g^{-1}x) = (x, \chi(g)F(g^{-1}x)).$$

Thus s is G-invariant if and only if $F(gx) = \chi(g)x$ for all $g \in G, x \in \mathbb{C}^r$. We denote the set of invariant sections by $H^0(\mathbb{C}^r, \mathcal{L}_\chi)^G$. A polynomial F as above is called a **semiinvariant** with weight χ.

For $x \in \mathbb{C}^r$ and $\chi \in \widehat{G}$ we say that x is **semistable** (with respect to χ) if there exists $d > 0$ and $s \in H^0(\mathbb{C}^r, \mathcal{L}_{\chi^d})^G$ with $s(x) \neq 0$. We denote the sets of semistable points by $(\mathbb{C}^r)^{ss}_\chi$.

We say that x is **stable** (with respect to χ) if x is semistable, its orbit $Gx \subset (\mathbb{C}^r)^{ss}_\chi$ is closed in $(\mathbb{C}^r)^{ss}_\chi$ and the stabilizer $G_x = \{g \in G; gx = x\}$ of x is finite. We denote the sets of stable points by $(\mathbb{C}^r)^s_\chi$.

One of our main goals in the following sections will be to analyze how these sets change, when we vary $\chi \in \widehat{G}$.

Before we proceed with the general theory, we look at how the above definitions relate to toric varieties. For the variety X_Σ we have the group $G = \mathrm{Hom}(\mathrm{Cl}(X_\Sigma), \mathbb{C}^*)$ acting on $\mathbb{C}^{\Sigma(1)}$ in the way described above. We observe that there is a canonical map

$$\mathrm{Cl}(X_\Sigma) \to \widehat{G}, \beta \mapsto (\varphi \mapsto \varphi(\beta)),$$

which is an isomorphism as $\mathrm{Cl}(X_\Sigma)$ is finitely generated and abelian. Moreover we want to mention that by Proposition 4.2.5 and 4.2.6 of [CLS11] the Class group is free if Σ contains a full-dimensional cone and X_Σ is smooth. In this case G is actually itself a torus.

In Section 2.3.4 we saw that X_Σ can be recovered by taking the quotient of a suitable subset of $\mathbb{C}^{\Sigma(1)}$ by G. We will now see that this subset is exactly the set of semistable points for a suitable character.

Theorem 3.1 (see Proposition 14.1.9 in [CLS11])**.** Let X_Σ be a projective toric variety corresponding to a fan Σ in $N_\mathbb{R}$. Let $\Sigma' \subset \Sigma$ be the subfan of simplicial cones in Σ. Let $\beta \in \mathrm{Cl}(X_\Sigma)$ be an ample divisor class coming from a character $\chi \in \widehat{G}$. Then

$$\left(\mathbb{C}^{\Sigma(1)}\right)^{ss}_\chi = \mathbb{C}^{\Sigma(1)} \backslash Z(\Sigma)$$
$$\left(\mathbb{C}^{\Sigma(1)}\right)^{s}_\chi = \mathbb{C}^{\Sigma(1)} \backslash Z(\Sigma')$$

We remark that as X_Σ is projective, there always exists an ample divisor β. We will first develop more theory and prove the first part of the Theorem using Corollary 3.13 below.

Back in the general setting we now want to define the quotient of $\mathbb{C}^r$ by the action of G. For every character $\chi \in \widehat{G}$ we define the ring

$$R_\chi = \bigoplus_{d=0}^{\infty} H^0(\mathbb{C}^r, \mathcal{L}_{\chi^d})^G \tag{3.1}$$

with the natural grading by $\deg(s) = d$ for $s \in H^0(\mathbb{C}^r, \mathcal{L}_{\chi^d})$. We denote the graded piece of degree d by $(R_\chi)_d$. Then we have that R_χ is finitely generated by a famous result of Nagata (see [Nag63], where we use that the torus is geometrically reductive). For $G \subset (\mathbb{C}^*)^r$ and $\chi \in \widehat{G}$ we define the **projective quotient**

$$\mathbb{C}^r /\!\!/_\chi G = \mathrm{Proj}(R_\chi).$$

Below you find some general properties of such quotients.

Proposition 3.2 (see Proposition 14.1.12 in [CLS11])**.** For $G \subset (\mathbb{C}^*)^r$ and $\chi \in \widehat{G}$ we have

1. There is a projective morphism

$$\mathbb{C}^r /\!\!/_\chi G \to \mathbb{C}^r /\!\!/ G = \mathrm{Spec}(\mathbb{C}[x_1, \ldots, x_r]^G) = \mathrm{Spec}((R_\chi)_0)$$

   defined by sending a homogeneous prime ideal $p \subset R_\chi$ to $p \cap (R_\chi)_0$.

2. $\mathbb{C}^r /\!\!/_\chi G \neq \varnothing$ if and only if $(\mathbb{C}^r)^{ss}_\chi \neq \varnothing$.

3. There is a projection map $(\mathbb{C}^r)^{ss}_\chi \to \mathbb{C}^r /\!\!/_\chi G$ induced by $x \mapsto m_x$ where m_x is the ideal generated by all invariant sections $s \in H^0(\mathbb{C}^r, \mathcal{L}_{\chi^d})$ which vanish at x for all $d \geq 0$. This is a well defined morphism and a good categorical quotient of $(\mathbb{C}^r)^{ss}_\chi$. Hence $\mathbb{C}^r /\!\!/_\chi G \cong (\mathbb{C}^r)^{ss}_\chi /\!\!/ G$, which shows that $\mathbb{C}^r /\!\!/_\chi G$ depends, up to isomorphism, only on the semistable set of χ.

4. If there exist stable points with respect to χ then the above quotient is almost geometric and $\dim \mathbb{C}^r /\!\!/_\chi G = r - \dim G$.

Using this we have the following Corollary of Theorem 3.1.

Corollary 3.3. Let X_Σ be a projective toric variety of the fan $\Sigma \subset N_\mathbb{R}$. Let $\beta \in \mathrm{Cl}(X_\Sigma)$ be an ample divisor class coming from a character $\chi \in \widehat{G}$. Then we have $X_\Sigma \cong \mathbb{C}^r /\!\!/_\chi G$.

Proof. By the proposition above we have $\mathbb{C}^r /\!\!/_\chi G \cong (\mathbb{C}^r)^{ss}_\chi /\!\!/ G$. By Theorem 3.1 we know that $\left(\mathbb{C}^{\Sigma(1)}\right)^{ss}_\chi = \mathbb{C}^{\Sigma(1)} \backslash Z(\Sigma)$. Now as X_Σ is projective, the fan Σ is complete by Theorem 3.1.19 of [CLS11] and in particular it is full-dimensional. Hence X_Σ has no torus factor. But by the discussion in Section 2.3.4 (or explicitly Theorem 5.1.11. in [CLS11]), we have $(\mathbb{C}^{\Sigma(1)} \backslash Z(\Sigma)) /\!\!/ G \cong X_\Sigma$. □

3.1.3. Toric group actions and Gale duality

The goal of this section is to analyse the quotients $(\mathbb{C}^r) /\!\!/_\chi G$ we defined in the previous section. Remember that we have a closed subgroup $G \subset (\mathbb{C}^*)^r$ acting on $\mathbb{C}^r$ via the restriction of the diagonal action $(\mathbb{C}^*)^r \curvearrowright \mathbb{C}^r$. In Theorem 3.9 we will see that the quotients are toric varieties coming from a generalized fan. This generalized fan in turn can be computed from the corresponding character χ.
First we have to look a little closer at the group G. In (2.1) we saw that given a

toric variety X_Σ without torus factor we obtain an exact sequence involving M and the character group $\widehat{G} = \mathrm{Cl}(X_\Sigma)$. This also holds in the more general case we consider now. When applying the functor $\mathrm{Hom}(-,\mathbb{C}^*)$ to the inclusion $G \hookrightarrow (\mathbb{C}^*)^r$, we obtain a map $\gamma : \mathbb{Z}^r \to \widehat{G}$. Let M denote the kernel of this map, then we have an exact sequence

$$0 \to M \xrightarrow{\delta} \mathbb{Z}^r \xrightarrow{\gamma} \widehat{G}. \tag{3.2}$$

Let $\nu_i = \delta^*(e_i) \in N$ so that $\delta(m) = (\langle m,\nu_1\rangle,\dots,\langle m,\nu_m\rangle)$. We also denote $\chi^a = \gamma(a)$ for $a \in \mathbb{Z}^r$ and we set $\beta_i = \gamma(e_i) \in \widehat{G}$. We define the torus $T_N = N \otimes_\mathbb{Z} \mathbb{C}^* = \mathrm{Hom}(M,\mathbb{C}^*)$. Then we have the following result.

Lemma 3.4 (see Lemma 14.2.1 in [CLS11])**.** For $G \subset (\mathbb{C}^*)^r$ closed and maps δ,γ as above we have

1. The map γ is surjective. In particular every character $\chi : G \to \mathbb{C}^*$ has an extension $\widehat{\chi} : (\mathbb{C}^*)^r \to \mathbb{C}^*$ and we obtain an exact sequence

$$0 \to M \xrightarrow{\delta} \mathbb{Z}^r \xrightarrow{\gamma} \widehat{G} \to 0 \tag{3.3}$$

2. We have $(\mathbb{C}^*)^r/G \cong T_N$ and thus

$$1 \to G \to (\mathbb{C}^*)^r \to T_N \to 1. \tag{3.4}$$

3. We can characterize the elements of $(\mathbb{C}^*)^r$ sitting in G by

$$G = \left\{(t_1,\dots,t_r) \in (\mathbb{C}^*)^r; \prod_{i=1}^r t_i^{\langle m,\nu_i\rangle} = 1 \text{ for all } m \in M\right\}.$$

In order to study how the quotient $\mathbb{C}^r /\!\!/_\chi G$ changes when we vary χ, we will use the concept of Gale duality from linear algebra, which was introduced in Section 2.4. It turns out that many of the properties of the quotient, for instance whether there exist (semi-)stable points, can be obtained from two finite collections of vectors, which are Gale dual to one another.
We start by considering the exact sequence from (3.3). To obtain an exact sequence of finite dimensional vector spaces as in (2.3) we tensorize this sequence with $\mathbb{R}$. We denote as usual $M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R}$ and $\widehat{G}_\mathbb{R} = \widehat{G} \otimes_\mathbb{Z} \mathbb{R}$ with the corresponding maps $\delta_\mathbb{R} = \delta \otimes \mathrm{id}_\mathbb{R}, \gamma_\mathbb{R} = \gamma \otimes \mathrm{id}_\mathbb{R}$ and we get the short exact sequence

$$0 \to M_\mathbb{R} \xrightarrow{\delta_\mathbb{R}} \mathbb{R}^r \xrightarrow{\gamma_\mathbb{R}} \widehat{G}_\mathbb{R} \to 0. \tag{3.5}$$

Then as described in Section 2.4 this sequence gives two collections of vectors $\boldsymbol{\beta} \in (\widehat{G}_\mathbb{R})^r$ and $\boldsymbol{\nu} \in (M_\mathbb{R}^*)^r = N_\mathbb{R}^r$ which are Gale dual to one another. Here as usual N is the dual of M and $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$. We denote by

$$C_\nu = \mathrm{Cone}(\boldsymbol{\nu}) \subset N_\mathbb{R} \quad \text{and} \quad C_\beta = \mathrm{Cone}(\boldsymbol{\beta}) \subset \widehat{G}_R$$

the cones spanned by those sets.
To get a first feeling for the vectors in $\boldsymbol{\beta}$, we look at the special case where the group G is torsion free. Then as seen above, it is actually a torus $G = (\mathbb{C}^*)^s$ and by general theory, the inclusion $G \hookrightarrow (\mathbb{C}^*)^r$ has the form

$$(t_1, \dots, t_s) \mapsto \left(t_1^{c_1^1} t_2^{c_1^2} \dots t_s^{c_1^s}, \dots, \prod_{j=1}^{s} t_j^{c_i^j}, \dots, t_1^{c_r^1} t_2^{c_r^2} \dots t_s^{c_r^s} \right)$$

for vectors $c_i = (c_i^j)_{j=1}^s \in \mathbb{Z}^s$ for $i = 1, \dots, r$. But when we take the dual map $\delta : \mathbb{Z}^r \to \mathbb{Z}^s = \widehat{(\mathbb{C}^*)^s}$, it will send the basis vector $e_i \in \mathbb{Z}^r$ to the character $(t_1, \dots, t_s) \mapsto \prod_{j=1}^s t_j^{c_i^j}$ corresponding to $c_i \in \mathbb{Z}^s$. But then obviously $c_i = \beta_i$, so for G torsion free, the vectors in $\boldsymbol{\beta}$ give the exponents of the inclusion map of G in $(\mathbb{C}^*)^r$.
To explore further the meaning of $\boldsymbol{\beta}$ and $\boldsymbol{\nu}$ we return to the representation of toric varieties as quotients studied in Section 2.3.4. There we had the toric variety X_Σ of a fan Σ in $N_\mathbb{R}$ and we considered the action of the group $G = \mathrm{Hom}(\mathrm{Cl}(X_\Sigma), \mathbb{C}^*)$ on $\mathbb{C}^{\Sigma(1)}$. Then the exact sequence (3.5) is obtained (via tensoring with $\mathbb{R}$) from the exact sequence

$$0 \to M \to \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z} D_\rho \to \mathrm{Cl}(X_\Sigma) \to 0, \tag{3.6}$$

seen in (2.1). Here the first map takes $m \in M$ to $\mathrm{div}(\chi^m) = \sum_{\rho \in \Sigma(1)} \langle m, u_\sigma \rangle D_\rho$ and the second map takes D_ρ to its equivalence class in $\mathrm{Cl}(X_\Sigma)$. Then it is immediate that

- $\boldsymbol{\beta} = ([D_\rho])_{\rho \in \Sigma(1)}$ and so C_β is the cone of torus-invariant effective divisor classes (with real coefficients) of X_Σ
- $\boldsymbol{\nu} = (u_\rho)_{\rho \in \Sigma(1)} \subset N_\mathbb{R}$ is the collection of primitive generators of the rays $\rho \in \Sigma(1)$ of the fan Σ and C_ν is the support of Σ

The intuition you should keep in mind during the subsequent sections is the following: the vectors $\boldsymbol{\beta}$ lie in $\widehat{G}_\mathbb{R}$ and they enable us to study how the quotient $\mathbb{C}^r /\!\!/_\chi G$ varies with $\chi \in \widehat{G}$. We will see that they partition the space $\widehat{G}_\mathbb{R}$ in a very intuitive and geometric way into pieces, where the quotient does not change. On the other hand, for a given $\chi \in \widehat{G}$ we will see in Theorem 3.9 that the quotient $\mathbb{C}^r /\!\!/_\chi G$ is a toric variety and that the rays of the fan corresponding to this variety can be chosen to be generated by some of the vectors in $\boldsymbol{\nu}$.

3.2. From characters to polyhedra and fans

In Section 3.3 we will see that the quotient $\mathbb{C}^r /\!\!/_\chi G$ is a toric variety. In the following we will show how to obtain a fan defining this variety from the character χ. In Section 3.2.1 we will do this by associating to χ a polyhedron and taking its normal fan. This is also the approach presented in [CLS11]. In Section 3.2.2 we will take a more direct approach and define the fan using so-called support functions.

3.2.1. From characters to polyhedra

Let $\chi = \chi^a$ for some $a \in \mathbb{Z}^r$, then we define

$$P_a = \{m \in M_\mathbb{R}; \langle m, \nu_i \rangle \geq -a_i \text{ for } i = 1, \ldots, r\}. \tag{3.7}$$

This is obviously a polyhedron. Note that different choices of a giving $\chi = \chi^a$ differ by elements in $\delta(M)$ and correspond to translates of the polyhedron: $P_{a+\delta(m)} = P_a - m$. As $\delta_\mathbb{R}$ is linear and injective, the set $\delta_\mathbb{R}(P_a) \subset \mathbb{R}^r$ is another polyhedron, isomorphic to P_a and a point $x \in \mathbb{R}^r$ is in $\delta_\mathbb{R}(P_a)$ if and only if it satisfies $x_i \geq -a_i$ and $\gamma_\mathbb{R}(x) = 0$. Noting that $\gamma_\mathbb{R}(a) = \chi \otimes 1$ we get an intrinsically defined polyhedron P_χ associated to $\chi \in \widehat{G}$ by

$$P_\chi = \{x \in \mathbb{R}^r; x_i \geq 0, \gamma_\mathbb{R}(x) = \chi \otimes 1\} = \delta_\mathbb{R}(P_a) + a \tag{3.8}$$

for all $a \in \mathbb{R}^r$ with $\gamma(a) = \chi$.
Now for all $i \in \{1, \ldots, r\}$ we define the i-th **virtual facet** of P_a as

$$F_{i,a} = \{m \in P_a; \langle m, \nu_i \rangle = -a_i\}.$$

From this representation it is clear that this is a facet of P_a, but it may well be empty. Looking at the correspondence between P_a and the more intrinsic polyhedron P_χ, we see that the corresponding facet of P_χ is defined by

$$F_{i,\chi} = P_\chi \cap V(x_i).$$

These facets will be important in analysing the semistable locus of our quotients.
In the definitions that follow there are possible complications if the polyhedron P_a is empty. Fortunately, the characters χ^a for which this happens can easily be identified.

Proposition 3.5. Let $a \in \mathbb{Z}^r$. Then the polyhedron P_a is nonempty iff $a \in \gamma_\mathbb{R}^{-1}(C_\beta)$ iff $\chi^a \otimes 1 \in C_\beta$.

Proof. The polyhedron P_a is nonempty if and only if there exists $m \in M_\mathbb{R}$ with $(\langle m, \nu_i \rangle + a_i)_{i=1}^r = \delta_\mathbb{R}(m) + a \in \mathbb{R}^r_{\geq 0}$. On the other hand $C_\beta = \gamma_\mathbb{R}(\mathbb{R}^r_{\geq 0})$, so $a \in \gamma_\mathbb{R}^{-1}(C_\beta)$ if and only if $a + \ker(\gamma_\mathbb{R}) \cap \mathbb{R}^r_{\geq 0} \neq \varnothing$. The fact that $\ker(\gamma_\mathbb{R}) = \operatorname{im}(\delta_\mathbb{R})$ shows the first equivalence. The second follows as $\chi^a \otimes 1 = \gamma_\mathbb{R}(a)$. □

From now on assume $a \in \gamma_\mathbb{R}^{-1}(C_\beta)$.
When P_a is given as above, its recession cone has the form

$$C = \{m \in M_\mathbb{R}; \langle m, \nu_i \rangle \geq 0 \text{ for } i = 1, \ldots, r\}.$$

Hence $C \cap (-C)$ is the set of $m \in M_\mathbb{R}$ with $\langle m, \nu_i \rangle = \delta_\mathbb{R}(m)_i = 0$ for all i. As $\delta_\mathbb{R}$ is injective, we have $C \cap (-C) = \{0\}$ and thus C is strongly convex. Looking again at the definition of C we can actually write it as the dual cone of the cone C_ν spanned by the vectors $\boldsymbol{\nu} \subset N_\mathbb{R}$. Its strong convexity then also follows from the general fact, that the dual cone of a full-dimensional cone is strongly convex.
Using the construction of a normal fan for a polyhedron with strongly convex rational

recession cone, we obtain a generalized fan $\Sigma_a = \Sigma_{P_a}$ for every $a \in \gamma_{\mathbb{R}}^{-1}(C_\beta)$. In fact even for general $a \in \mathbb{R}^r$ the polyhedron $P_a \subset M_{\mathbb{R}}$ can be defined using the formula (3.7) and for $a \in \gamma_{\mathbb{R}}^{-1}(C_\beta)$ we obtain the corresponding fan Σ_{P_a} as before.

3.2.2. From characters to fans

In the next section it will be useful to work with the polyhedra P_a and P_χ. However for studying the dependence of the quotients with respect to the character χ, we will work with fans rather than polyhedra. Therefore, it is interesting to see how to obtain the fan Σ_a directly from the vector $a \in \mathbb{R}^r$. The following constructions and statements are essentially the content of Definition 4.2.11, Definition 6.1.12, Theorem 7.2.2, Exercise 6.1.10 and Proposition 14.4.1. from [CLS11]. However, we elaborate the theory a bit more and take a more general approach, for instance in Lemma 3.6.
Given a generalized fan $\Sigma \subset N_{\mathbb{R}}$ a function $\varphi : |\Sigma| \to \mathbb{R}$ is called a **support function** with respect to Σ if it is linear on all cones of Σ. That is for all $\sigma \in \Sigma$ we have $\varphi|_\sigma$ is the restriction of a linear function $N_{\mathbb{R}} \to \mathbb{R}$. The vector space of support functions on Σ is denoted by $\mathrm{SF}(\Sigma)$.
For a function $\varphi : C \to \mathbb{R}$ defined on a convex set $C \subset N_{\mathbb{R}}$ we say that φ is **convex** if

$$\varphi(tx + (1-t)y) \geq t\varphi(x) + (1-t)\varphi(y) \text{ for all } t \in (0,1), x, y \in |\Sigma|.$$

Note that this definition differs from the more customary definition, where the inequality is reversed. In our convention, a function φ is convex if the set of points under the graph $\{(x, \varphi(x)); x \in |\Sigma| \times \mathbb{R}\}$ of φ is convex. The set of convex support functions on Σ is denoted by $\mathrm{CSF}(\Sigma)$.
Now assume that the generalized fan Σ is full-dimensional (i.e. the dimension of the maximal cones of Σ is equal to $n = \dim N_{\mathbb{R}}$) with convex support $|\Sigma|$ and let $\varphi \in \mathrm{CSF}(\Sigma)$. Then for any maximal cone $\sigma \in \Sigma(n)$ there exists a unique $m_\sigma \in M_{\mathbb{R}} = (N_{\mathbb{R}})^*$ with

$$\varphi(u) = \langle m_\sigma, u \rangle \text{ for } u \in \sigma. \tag{3.9}$$

We say that the function φ is **strictly convex** if for $u \in |\Sigma|, \sigma \in \Sigma(n)$ we have

$$u \in \sigma \iff \varphi(u) = \langle m_\sigma, u \rangle.$$

One can prove (see Lemma 6.1.13 in [CLS11]) that φ is strictly convex iff it is convex and $m_\sigma \neq m_{\sigma'}$ for $\sigma, \sigma' \in \Sigma(n)$ with $\sigma \neq \sigma'$. Note that the convexity of φ together with (3.9) implies that

$$\varphi(u) = \min(\langle m_\sigma, u \rangle; \sigma \in \Sigma(n)).$$

Now we want to find a way to recover the generalized fan Σ only given a strictly convex support function on Σ. This is indeed possible, but even more is true.
For a finite set $S \subset M_{\mathbb{R}}$ and a polyhedral cone $C \subset N_{\mathbb{R}}$ we define a function

$$\varphi_S : C \to \mathbb{R}, u \mapsto \min(\langle m, u \rangle; m \in S).$$

It is easy to see that the function φ_S is convex on C. Furthermore, for any generalized fan $\Sigma \subset N_{\mathbb{R}}$ with $|\Sigma| = C$, every convex support function on Σ is of the form φ_S, where $S = \{m_\sigma; \sigma \in \Sigma(n)\}$ as seen above. Now we show a converse to this.

Lemma 3.6. Let $S \subset M_{\mathbb{R}}$ finite, $C \subset N_{\mathbb{R}}$ a full-dimensional polyhedral cone. Then there exists a unique generalized fan Σ in $N_{\mathbb{R}}$ with $|\Sigma| = C$, such that φ_S is a strictly convex support function on Σ.

Proof. For every nonempty subset $S' \subset S$ we define the set

$$\sigma_{S'} = \{u \in C; \varphi_S(u) = \langle m, u\rangle \text{ for } m \in S'\} \subset C.$$

We claim that these sets σ_m are polyhedral cones and that, together with their faces, they form the desired generalized fan $\Sigma = \{\tau; \tau \preceq \sigma_{S'}, \varnothing \neq S' \subset S\}$. For this observe that $\sigma_{S'}$ is exactly cut out from C by the finitely many linear inequalities

$$\langle m, u\rangle \geq \langle m', u\rangle \text{ for } m \in S, m' \in S'.$$

This shows that the $\sigma_{S'}$ are indeed polyhedral cones and they cover C. Moreover it is clear that $\sigma_{S'} \cap \sigma_{S''} = \sigma_{S' \cup S''}$ and we claim that for $S' \subset S''$ we have $\sigma_{S''} \preceq \sigma_{S'}$. To see this last point we may reduce to the case $S'' = S' \cup \{\overline{m}\}$ for some $\overline{m} \in S$. Then on $\sigma_{S'}$ we already have

$$\langle m, \cdot\rangle = \varphi_S(\cdot), \langle \overline{m} - m, \cdot\rangle \geq 0 \text{ for } m \in S'.$$

We certainly have $\sigma_{S''} \subset \sigma_{S'}$. Thus for some $m \in S'$ the linear function $\langle \overline{m} - m, \cdot\rangle$ is nonnegative on $\sigma_{S'}$ and it vanishes exactly on $\sigma_{S''}$, hence $\sigma_{S''} \preceq \sigma_{S'}$.
From the above observations it follows that indeed Σ is a generalized fan with $|\Sigma| = C$ and it is clear that φ_S is a convex support function on Σ. To see that φ_S is strictly convex on Σ observe that any maximal dimensional cone in Σ must be of the form $\sigma_{S'}$ for $\varnothing \neq S' \subset S$. But $m_1, m_2 \in S'$ implies $\langle m_1, \cdot\rangle = \langle m_2, \cdot\rangle = \varphi_S$ on $\sigma_{S'}$ and since this set is full-dimensional it follows $m_1 = m_2$. Hence $S' = \{m\}$ and this single element m is uniquely determined by $\sigma_{S'}$, so that distinct maximal cones correspond to different elements $m \in S$. This shows the criterion for strict convexity mentioned above.
For the uniqueness we assume that Σ' is another generalized fan making φ_S a strictly convex function. For every $\sigma' \in \Sigma'(n)$ we have that $\langle m_{\sigma'}, \cdot\rangle = \min(\langle m, u\rangle; m \in S)$. So at every point u of σ' one of the equalities $\langle m_{\sigma'}, u\rangle = \langle m, u\rangle$ for $m \in S$ has to hold. But unless $m_{\sigma'} \in S$, the sets where these equalities hold are all of codimension 1. Thus they could not cover the full-dimensional set σ'. This implies $\{m_{\sigma'}; \sigma' \in \Sigma'(n)\} \subset S$. But now the condition of strict convexity on φ_S tells us that $\sigma' = \{u \in C; \varphi_S(u) = \langle m_{\sigma'}, u\rangle\} = \sigma_{\{m_{\sigma'}\}} \in \Sigma(n)$. This shows that $\Sigma'(n) \subset \Sigma(n)$ and as $|\Sigma| = |\Sigma'| = C$ we obtain $\Sigma = \Sigma'$. □

Note here that if we add a linear function $\langle m, \cdot\rangle$ to φ_S the unique fan Σ from above does not change. Furthermore it follows from the proof that if the set $S \subset M_{\mathbb{R}}$ defining the function φ_S is minimal among the subsets of $M_{\mathbb{R}}$ giving this function, then $S = \{m_{\sigma'}; \sigma' \in \Sigma(n)\}$, where Σ is the fan from the proof. Conversely, the maximal cones $\sigma' \in \Sigma(n)$ are then exactly the cones $\sigma_{\{m\}} = \{u \in C; \varphi_S(u) = \langle m, u\rangle\}$ for $m \in S$.

In Section 3.4 we will look at the map associating to vectors $a \in \mathbb{R}^r$ with $P_a \neq \emptyset$ the normal fan $\Sigma_a = \Sigma_{P_a}$ of P_a. We will see now, that this generalized fan can be obtained as the unique generalized fan making a certain function φ_a strictly convex.
Let $\nu_1, \dots, \nu_r \in N_{\mathbb{R}}$ be vectors, $C = \mathrm{Cone}(\nu_i; i = 1, \dots, r)$ and let $a = (a_1, \dots, a_r) \in \mathbb{R}^r$. Then we can look at the vectors

$$\tilde{\nu}_i = (\nu_i, -a_i) \in N_{\mathbb{R}} \times \mathbb{R}$$

and the cone $\tilde{C} \subset N_{\mathbb{R}} \times \mathbb{R}$ spanned by them. We want to define the function $\varphi_a : C \to \mathbb{R}$ such that its graph (as a subset of $C \times \mathbb{R}$) is exactly the union of the facets of $\tilde{C}$ with downward pointing inner normal vectors. We will refer to these facets as the upper facets in the text below.
In order for this to be well-defined, we will first restrict ourself to the case when $a \in \mathbb{R}^r_{\geq 0}$. Then clearly $\tilde{C} \subset N_{\mathbb{R}} \times \mathbb{R}_{\leq 0}$ and we define

$$\varphi_a(x) = \sup\left(h; (x,h) \in \tilde{C}\right) \text{ for } x \in C. \tag{3.10}$$

Writing this out more explicitly in terms of the data $\boldsymbol{\nu}, a$ we see

$$\varphi_a(x) = \sup\left(-\sum_{i=1}^{r} \lambda_i a_i; \lambda_1, \dots, \lambda_r \geq 0, \sum_{i=1}^{r} \lambda_i \nu_i = x\right).$$

Looking at the first description of φ_a a little argument shows, that φ_a is convex and actually the maximum of a finite set of linear functions. The graphs of these functions in $N_{\mathbb{R}} \times \mathbb{R}$ are the hyperplanes defined by the upper facets of $\tilde{C}$. We define the generalized fan $\Sigma(a)$ to be the unique generalized fan making φ_a a strictly convex function. Using the remark after the proof of Lemma 3.6 one sees that the maximal cones of $\Sigma(a)$ are exactly the projections of the upper facets of $\tilde{C}$ to C.
Another information, which will turn out to be interesting, is to determine, which of the vectors $\tilde{\nu}_i$ do not "contribute" to φ_a. To make this precise, we define

$$I_{\emptyset}(a) = \{i; \varphi_a(\nu_i) > -a_i\} \subset \{1, \dots, r\}.$$

Our goal in the remaining part of this section will be to give some intuition about how the fan $\Sigma(a)$ and the set of indices $I_{\emptyset}(a)$ depend on a. Finally we will show that $\Sigma(a) = \Sigma_a$ is the normal fan of the polyhedron P_a defined earlier.
First we note that if we add to $a \in \mathbb{R}^r_{\geq 0}$ a vector of the form $\delta_{\mathbb{R}}(m) = (\langle m, \nu_i \rangle)_{i=1}^r$ for $m \in M_{\mathbb{R}}$, the constructions above will still be well-defined and we see $\varphi_{a+\delta(m)} = \varphi_a - \langle m, \cdot \rangle$. Here $\delta_{\mathbb{R}}$ comes from the exact sequence (3.5), which we used in the context of Gale duality. Thus we can actually define φ_a, $\Sigma(a)$ and $I_{\emptyset}(a)$ for $a \in \mathbb{R}^r_{\geq 0} + \delta_{\mathbb{R}}(M) = \gamma_{\mathbb{R}}^{-1}(C_{\boldsymbol{\beta}})$. Moreover, we see that they can only depend on the image of a under the map $\gamma_{\mathbb{R}} : \mathbb{R}^r \to \widehat{G}_{\mathbb{R}}$.

Theorem 3.7. Let $\boldsymbol{\nu} = \{\nu_1, \dots, \nu_r\} \subset N_{\mathbb{R}}$ be a set of vectors, $\boldsymbol{\beta} = \{\beta_1, \dots, \beta_r\} \subset \widehat{G}_{\mathbb{R}}$ its Gale dual and $a \in \gamma_{\mathbb{R}}^{-1}(C_\beta) \subset \mathbb{R}^r$. Take the cone $C_\nu \subset N_{\mathbb{R}}$ spanned by the vectors in $\boldsymbol{\nu}$ as a convex set and define the function φ_a as above. Let $S \subset M_{\mathbb{R}}$ be the minimal set

defining φ_a and let $\Sigma(a)$ be the generalized fan with support C_ν corresponding to φ_a like in Lemma 3.6.
Then S is the set of vertices of the polyhedron P_a. Furthermore $\Sigma(a)$ is the normal fan of P_a and $I_\varnothing(a)$ is the set of indices corresponding to empty virtual facets, so

$$\Sigma(a) = \Sigma_a \quad \text{and} \quad I_\varnothing(a) = \{i; F_{i,a} = \varnothing\}.$$

Moreover, we have that the rays $\Sigma(a)(1)$ of $\Sigma(a)$ are spanned by vectors contained in the set $\{\nu_i; i \notin I_\varnothing(a)\} \subset \boldsymbol{\nu}$ and that

$$\sigma = \operatorname{Cone}(\nu_i; \nu_i \in \sigma, i \notin I_\varnothing(a)) \tag{3.11}$$

for all $\sigma \in \Sigma(a)$.

Proof. As seen in Proposition 3.5 the choice of $a \in \gamma_{\mathbb{R}}^{-1}(C_\beta)$ ensures that P_a is nonempty and $\Sigma(a), I_\varnothing(a)$ are defined. Recall the definition $P_a = \{m; \langle m, \nu_i\rangle \geq -a_i\}$. From this it follows immediately that $P_a \subset M_{\mathbb{R}} = (N_{\mathbb{R}})^*$ is exactly the set of linear functions on $N_{\mathbb{R}}$ which are greater or equal to φ_a. Hence $S \subset P_a$. Now any $m \in S$ corresponds to a maximal cone $\sigma \in \Sigma(a)$. Pick a point $u \in \operatorname{Int}(\sigma)$ and look at the half-space

$$H_{u,\varphi_a(u)} = \{m'; \langle m', u\rangle \geq \varphi_a(u)\} \subset M_{\mathbb{R}}.$$

It is obvious that it contains P_a (we only demand the linear function $\langle m', \cdot\rangle$ to be greater or equal to φ_a at one point u). But if for some $m' \in P_a$ we have $\langle m', u\rangle = \varphi_a(u)$ then we have $\langle m' - m, \cdot\rangle \geq 0$ on σ and equality at a point in the interior of σ, so $m' = m$. This shows that all elements m of S are vertices of P_a.
Conversely, any vertex m of P_a must be given as the unique minimum of $\langle \cdot, u\rangle$ for some $u \in N_{\mathbb{R}}$ on P_a. For $\langle \cdot, u\rangle$ to be bounded below on P_a we need it to be nonnegative on the recession cone C of P_a. This is the same as asking $u \in C^\vee = (C_\nu^\vee)^\vee = C_\nu$, where the equality $C = C_\nu^\vee$ was remarked towards the end of Section 3.2.1. But as a general fact, the set of $u \in N_{\mathbb{R}}$ giving some vertex of a polyhedron in above way is an open set. As the maximal cones of $\Sigma(a)$ cover C_ν we may thus choose the point u in the interior of one of them. But the same argument as above implies that m is one of the points in S (which are in bijection with the maximal cones). Hence S is the set of vertices of P_a.
Now let $m \in S$ be a vertex of P_a. From the definition of the normal fan it follows that its maximal cones are in bijection with the vertices of P_a, where m corresponds to the cone

$$\begin{aligned} C_m^\vee &= \operatorname{Cone}(P_a \cap M_{\mathbb{Q}} - m)^\vee \\ &= \{u \in N_{\mathbb{R}}; \langle m' - m, u\rangle \geq 0 \text{ for all } m' \in P_a\} \\ &= \{u \in N_{\mathbb{R}}; \langle m' - m, u\rangle \geq 0 \text{ for all } m' \in S\} \\ &= \sigma_{\{m\}} \in \Sigma(a)(u). \end{aligned}$$

This shows the statement about the normal fans.
For any index $i \in \{1, \dots, r\}$ we have that the virtual facet $F_{i,a}$ is empty if and only if $\langle m, \nu_i\rangle > -a_i$ for all $m \in P_a$. But for this it suffices checking on the vertices $m \in S$ of

P_a (as the recession cone C is the dual of the cone C_ν). But this in turn is equivalent to $\varphi_a(\nu_i) = \min_{m\in S}(\langle m, \nu_i\rangle) > -a_i$ i.e. $i \in I_\varnothing$.
For the two final statements it suffices to show the second, as the first follows by applying the second one to the rays in $\Sigma(a)$. First observe that in (3.11) the inclusion "$\supset$" is clear. Moreover if σ is a cone spanned by some set R and τ is a face of σ then τ is spanned by the elements of R contained in τ. One deduces that it suffices to check the inclusion "$\subset$" for the maximal cones of $\Sigma(a)$. Let $\sigma \in \Sigma(a)$ be a maximal cone, then it is the image of an upper facet F of the cone $\tilde{C} \subset N_\mathbb{R} \times \mathbb{R}$ under the projection to $N_\mathbb{R}$. Again as above F is spanned by those of the vectors $\tilde{\nu}_i = (\nu_i, -a_i)$ (which span $\tilde{C}$) contained in F. But $\tilde{\nu}_i \in F$ is equivalent to $\nu_i \in \sigma$ and $\varphi(\nu_i) = -a_i$, i.e. $i \notin I_\varnothing$. □

3.3. From polyhedra to quotients

The toric variety X_P of a nonempty polyhedron P with strongly convex rational recession cone is defined as the toric variety of its normal fan Σ_P. For convenience we say that the toric variety of the empty polyhedron is the empty set. We want to show that for $\chi = \chi^a \in \widehat{G}$ we have $\mathbb{C}^r /\!\!/_\chi G \cong X_{P_a}$. The following construction establishes one of the crucial links between the two objects.
For a polyhedron $P \subset M_\mathbb{R}$ as above we can define the cone $C(P) \subset M_\mathbb{R} \times \mathbb{R}$ over the polyhedron by

$$C(P) = \overline{\{(m,\lambda) \in M_\mathbb{R} \times \mathbb{R}; \lambda > 0, m \in \lambda P\}}. \tag{3.12}$$

It is immediate that $C(P) \cap \{\lambda = \lambda_0\} = \lambda_0 P \times \{\lambda_0\}$ for $\lambda_0 > 0$ and for P nonempty a little calculation shows $C(P) \cap \{\lambda = 0\} = C \times \{0\}$, where again C is the recession cone of P. Now $C(P) \cap M \times \mathbb{Z}$ is a semigroup and the corresponding algebra $S_P = \mathbb{C}[C(P) \cap M \times \mathbb{Z}]$ has a grading given by the projection $M \times \mathbb{Z} \to \mathbb{Z}$ on the second factor. Then the toric variety X_P has the following nice description.

Proposition 3.8 (see Proposition 14.2.12 in [CLS11])**.** For a polyhedron $P \subset M_\mathbb{R}$ with strongly convex rational recession cone we have: $X_P \cong \operatorname{Proj}(S_P)$.

Now we can state and prove the main result of this section.

Theorem 3.9 (see Theorem 14.2.13 in [CLS11])**.** Let $G \subset (\mathbb{C}^*)^r$ be a closed subgroup, $\chi = \chi^a \in \widehat{G}$ for some $a \in \mathbb{Z}^r$ then

1. The graded ring R_χ from (3.1) is isomorphic to $\mathbb{C}[C(P_a) \cap M \times \mathbb{Z}] = S_{P_a}$.
2. The quotient $\mathbb{C}^r /\!\!/_\chi G$ is isomorphic to the toric variety of P_a.

Proof. We first note that an element $x^b \in \mathbb{C}[x_1, \dots, x_r]$ gives a global section $s^b(x) = (x, x^b)$ of $\mathcal{L}$. Using the definitions in Section 3.1.3 one checks that s^b is a semiinvariant of weight χ iff $\gamma(b) = \chi$. Thus the monomials x^b with $\gamma(b) = d\chi$ correspond exactly to the sections $s^b \in H^0(\mathbb{C}^r, \mathcal{L}_{d\chi})^G$, which generate the G-invariant subspace $H^0(\mathbb{C}^r, \mathcal{L}_{d\chi})^G$ as a $\mathbb{C}$-vector space.
Thus for

$$S_\chi = \bigcup_{d=0}^{\infty} (\gamma^{-1}(\chi^d) \cap \mathbb{N}^d) \times \{d\} \subset \mathbb{N}^r \times \mathbb{N}$$

we get an isomorphism of $\mathbb{C}$-vector spaces $R_\chi \cong \mathbb{C}^{S_\chi}$. But S_χ inherits the structure of a semigroup from $\mathbb{N}^r \times \mathbb{N}$ by the linearity of γ (note that the group law of $\widehat{G}$ is written multiplicatively, i. e. $\chi^d \chi^e = \chi^{d+e}$). It is easy to see that this is compatible with the multiplication in R_χ, such that $R_\chi \cong \mathbb{C}[S_\chi]$ as a graded semigroup algebra.
Now we finish the proof of the first part of the theorem by showing that $C(P_a) \cap M \times \mathbb{Z} \cong S_\chi$. The isomorphism is given by $(m, d) \mapsto (\delta(m) + da, d)$. It is obviously enough to check this on the level sets of the second component and then you note that the equations defining P_{da} are exactly equivalent to $\delta(m) + da$ having nonnegative coordinates.
Now for the second part of the theorem we simply note that P_a satisfies the conditions of Proposition 3.8 and thus

$$\mathbb{C}^r /\!\!/_\chi G = \operatorname{Proj}(R_\chi) = \operatorname{Proj}(\mathbb{C}[C(P_a) \cap M \times \mathbb{Z}]) = X_{P_a}. \qquad \square$$

As $P_\chi \cong P_a$, we get the following immediate Corollary, which will be useful later.

Corollary 3.10. In the situation of Theorem 3.9 we have $\dim \mathbb{C}^r /\!\!/_\chi G = \dim X_{P_a} = \dim P_\chi$.

Now we can begin the study of how the quotient $\mathbb{C}^r /\!\!/_\chi G$ changes when varying χ. As we have $\mathbb{C}^r /\!\!/_\chi G \cong (\mathbb{C}^r)^{ss}_\chi /\!\!/ G$ it is clear that we have to analyze the set of semistable points. A basic question, namely if there exists such a semistable point, can be answered immediately.

Proposition 3.11 (see Proposition 14.3.5. in [CLS11])**.** For a character $\chi \in \widehat{G}$ we have that $(\mathbb{C}^r)^{ss}_\chi \neq \emptyset$ if and only if $\chi \otimes 1 \in C_{\boldsymbol{\beta}}$.

Proof. We saw above that the quotient $\mathbb{C}^r /\!\!/_\chi G$ is the toric variety of the polyhedron P_χ so there exists a semistable point iff this variety is nonempty iff $P_\chi \neq \emptyset$. But by Proposition 3.5 this is the case if and only if $\chi \otimes 1 \in C_\beta$. $\square$

This result shows that when looking for interesting quotients, we may restrict ourselves to characters inside the cone generated by the vectors in $\boldsymbol{\beta}$. One can then also show that there exists a stable point for the G-action with respect to a character χ if and only if $\chi \otimes 1$ is even contained in the interior of C_β and that in this case, the quotient $\mathbb{C}^r /\!\!/_\chi G$ has the expected dimension $r - \dim(G)$ (see Proposition 14.3.6 in [CLS11]).
To obtain an explicit description of the locus of semistable points, we use the notion of virtual facets defined earlier. The following result tells us how they are related to semistable points.

Proposition 3.12 (see Proposition 14.2.21 in [CLS11])**.** For a point $p \in \mathbb{C}^r$ let $I(p) = \{i \in \{1, \dots, r\}; p_i = 0\}$. Then $p \in (\mathbb{C}^r)^{ss}_\chi$ for some $\chi \in \widehat{G}$ if and only if $\bigcap_{i \in I(p)} F_{i,\chi} \neq \emptyset$. In particular for an index i with $F_{i,\chi} = \emptyset$ we have $(\mathbb{C}^r)^{ss}_\chi \subset D(x_i) = \{p \in \mathbb{C}^r; p_i \neq 0\}$.

Proof. A point $p \in \mathbb{C}^r$ is semistable with respect to χ if and only if there exists $s \in H^0(\mathbb{C}^r, \mathcal{L}_{\chi^d})^G$ with $s(p) \neq 0$. But as seen above these semiinvariant sections are generated by the sections corresponding to x^b for $b \in \mathbb{N}^r$ with $\gamma(b) = \chi$. Thus in

particular $b \in P_\chi \cap \mathbb{N}^r$ and $p^b \neq 0$ is equivalent to $b_i = 0$ for all $i \in I(p)$. But such a b is contained in the intersection from the theorem, which proves one direction.
For the other direction note that if the intersection $\bigcap_{i\in I(P)} F_{i,\chi}$ is nonempty, it has a rational point. Multiplying with a sufficiently large $d > 0$ we find a point $b \in \bigcap_{i\in I(P)} dF_{i,\chi} \cap \mathbb{Z}^r$. Then $p^b \neq 0$ and $\gamma_\mathbb{R}(b) = \chi^d \otimes 1$. Thus $\gamma(b)$ and χ^d differ by a torsion element of G and hence $\gamma(lb) = \chi^{dl}$ for $l > 0$ sufficiently large. Then x^{lb} gives a section in $H^0(\mathbb{C}^r, \mathcal{L}_{\chi^d})^G$ not vanishing at p. □

Now we have a possibility to give a description of the set of semistable points by using computations on the polyhedra. We define the **irrelevant ideal**

$$B(\chi) = \left(\prod_{i\notin I} x_i ; I \subset 1, \dots, r, \bigcap_{i\in I} F_{i,\chi} \neq \emptyset \right) \subset \mathbb{C}[x_1, \dots, x_r]$$

and its vanishing set $Z(\chi) = V(B(\chi)) \subset \mathbb{C}^r$. Then one sees easily from Proposition 3.12 that $Z(\chi)$ is exactly the complement of the semistable points with respect to χ. This gives the following result.

Corollary 3.13. $(\mathbb{C}^r)^{ss}_\chi = \mathbb{C}^r \backslash Z(\chi)$ and thus $\mathbb{C}^r /\!\!/_\chi G \cong (\mathbb{C}^r \backslash Z(\chi)) /\!\!/ G$.

Now we are in the position to prove the first part of Theorem 3.1.

Proof of Theorem 3.1. Remember that we had a projective toric variety X_Σ and we claimed that for the action of $G = \mathrm{Hom}(\mathrm{Cl}(X_\Sigma), \mathbb{C}^*)$ on $\mathbb{C}^{\Sigma(1)}$ and a character $\chi \in \widehat{G}$ corresponding to an ample divisor class $D = \sum_{\rho\in\Sigma(1)} a_\rho D_\rho$ we have

$$\left(\mathbb{C}^{\Sigma(1)}\right)^{ss}_\chi = \mathbb{C}^{\Sigma(1)} \backslash Z(\Sigma).$$

Now by Theorem 6.1.14 in [CLS11] the fact that D is ample is equivalent to saying that $\varphi_{(a_\rho)_{\rho\in\Sigma(1)}}$ is strictly convex on Σ. But then by definition $\Sigma = \Sigma(a)$ and it is the normal fan of the polyhedron P_a.
Remember that we defined $Z(\Sigma)$ as the vanishing set of $B(\Sigma) = (\prod_{\rho\notin\sigma(1)} x_\rho ; \sigma \in \Sigma) \subset \mathbb{C}[x_\rho ; \rho \in \Sigma(1)]$. Comparing with the definition of $B(\chi)$ above we see that it suffices to show that the sets $I = \sigma(1) \subset \Sigma(1)$ are exactly those subsets of $\Sigma(1)$ such that $\bigcap_{i\in I} F_{i,\chi} \neq \emptyset$. It is also sufficient to consider only the maximal such cones or sets, respectively, as they give minimal generators of the corresponding ideals.
Given a maximal cone $\sigma \in \Sigma$ we have that σ is n-dimensional since $|\Sigma| = N_\mathbb{R}$. Here we use that projective varieties correspond to complete fans by Theorem 3.4.6 in [CLS11]. The completeness of Σ also implies that the polyhedron P_a we defined above is actually a lattice polytope. To see this we observe that the recession cone of P_a is given by $C^\vee_\nu = N^\vee_\mathbb{R} = \{0\}$.
Now we know from the construction of the normal fan that the maximal cones σ in Σ correspond to the vertices v of P_a and the rays $\rho_i \in \sigma(1)$ correspond to the facets $F_{i,a}$ containing v. Conversely if I is maximal such that $\bigcap_{i\in I} F_{i,\chi} \neq \emptyset$, then the intersection is a vertex of P_χ. This shows the desired bijection. □

3.4. The secondary fan

We are now ready to define the main subject of our studies: the secondary fan Σ_{GKZ}. This rational polyhedral fan sits inside $\widehat{G}_{\mathbb{R}}$ with support C_β and for every cone $\sigma \in \Sigma_{\mathrm{GKZ}}$ we have that $\mathbb{C}^r /\!\!/_\chi G$ is essentially the same for all $\chi \otimes 1 \in \sigma$.
We will mostly use this fan and the related concepts when applied to concrete examples, where the geometric meaning of the cones and the corresponding quotients will be obvious. Therefore we will not give rigorous proofs of all the statements below and refer the interested reader to the proofs given in Section 14.4 of [CLS11]. Instead we try to give a good intuitive grasp on how to compute the secondary fan and the relation between its geometry and the geometry of the quotient varieties.

3.4.1. Preliminary remarks and observations

As we have seen in Section 3.3, the quotient $\mathbb{C}^r /\!\!/_{\chi^a} G$ is the toric variety of the polyhedron P_a and the corresponding normal fan $\Sigma(a)$. This reduces the problem how the quotient changes to the problem, how this fan changes. If moreover we want to track the information of the semistable set $(\mathbb{C}^r)^{ss}_\chi$, we have to take into consideration the set $I_\varnothing(a)$ as well.
To see this remember, that in Corollary 3.13 we identified the set of unstable points for the character χ^a as the vanishing locus of the ideal

$$B(\chi^a) = \left(\prod_{i \notin I} x_i ; I \subset 1, \dots, r, \bigcap_{i \in I} F_{i,\chi^a} \neq \varnothing \right) \subset \mathbb{C}[x_1, \dots, x_r].$$

Let $\mathcal{I} = \{I \subset \{1, \dots, r\}; F_I = \bigcap_{i \in I} F_{i,a} \neq \varnothing\}$. It is clear that the minimal generators of the ideal $B(\chi^a)$ correspond to the maximal sets $I \in \mathcal{I}$. But for such I maximal, F_I is a face of P_a. If it is not a vertex then there exists a facet $F_{j,\chi} \preceq P_a$ such that $F_I \cap F_{j,\chi}$ is nonempty and strictly contained in F_I. Here we use the fact, that the recession cone of P_a is strongly convex. But this would imply $I \cup \{j\} \in \mathcal{I}$, a contradiction.
Thus the minimal generators of $B(\chi^a)$ correspond to the vertices of P_a. For every vertex m of P_a we have the monomial corresponding to the maximal set $I \subset \{1, \dots, r\}$ with $F_I = \{m\}$, namely $I = \{i; m \in F_{i,a}\}$. But the vertices of P_a are in turn bijective to the maximal cones $\sigma \in \Sigma(a)$. Let m correspond to such a maximal cone σ, then for all i we have

$$m \in F_{i,a} \iff \langle m, \nu_i \rangle = -a_i \iff \langle m, \nu_i \rangle = \varphi_a(\nu_i) = -a_i \iff \nu_i \in \sigma \text{ and } i \notin I_\varnothing(a).$$

Hence the semistable set only depends on $\Sigma = \Sigma(a)$ and $I_\varnothing = I_\varnothing(a)$ and the vanishing ideal of its complement is defined by

$$B(\Sigma, I_\varnothing) = \left(\prod_{\nu_i \notin \sigma \text{ or } i \in I_\varnothing} x_i ; \sigma \in \Sigma_{\max} \right). \tag{3.13}$$

Now in Theorem 3.7 we saw that for all $a \in \gamma_{\mathbb{R}}^{-1}(C_\beta)$ the rays of $\Sigma(a)$ are contained in the finite set $\{\mathrm{Cone}(\nu_i); i = 1, \dots, r\}$. Thus any possible fan $\Sigma(a)$ can be uniquely described by the set

$$\{\{i; \nu_i \in \sigma\}; \sigma \in \Sigma(a)\} \subset 2^{\{1,\dots,r\}}.$$

This gives the rather coarse but finite bound of 2^{2^r+r} on the number of possible pairs $(\Sigma, I_\varnothing)$ that can occur as data for our quotient.
A pair $(\Sigma, I_\varnothing)$ of a generalized fan Σ in $N_{\mathbb{R}}$ and a set of indices $I_\varnothing \subset \{1, \dots, r\}$ coming from some $a \in \mathbb{R}^r$ as described above will be called an **admissible pair**. This is no standard terminology, but as they will appear quite often, this convention will be useful. Let now $(\Sigma, I_\varnothing)$ be an admissible pair. Using the results of Theorem 3.7 it is then easy to see, that the set of $a \in \mathbb{R}^r$ such that $\Sigma = \Sigma(a), I_\varnothing = I_\varnothing(a)$ is exactly

$$\tilde{\Gamma}^0_{\Sigma,I_\varnothing} = \left\{ a \in \gamma_{\mathbb{R}}^{-1}(C_\beta); \varphi_a \text{ strictly convex on } \Sigma \text{ and } \varphi_a(\nu_i) \begin{cases} = -a_i & \text{for } i \notin I_\varnothing \\ > -a_i & \text{for } i \in I_\varnothing \end{cases} \right\}.$$

We now want to analyze the structure of these sets. Firstly we note that, by definition, they form a disjoint system of sets with union $\gamma_{\mathbb{R}}^{-1}(C_\beta)$. Also it is clear that they are invariant under translation by vectors in $\delta_{\mathbb{R}}(M) = \ker(\gamma_{\mathbb{R}})$. Finally, a short argument shows that they are convex and invariant under scaling with a positive factor. However, they are in general not polyhedral cones, but rather the relative interior of such cones.

3.4.2. Definition and first properties

Let $(\Sigma, I_\varnothing)$ be an admissible pair. Then we define

$$\tilde{\Gamma}_{\Sigma,I_\varnothing} = \left\{ a \in \gamma_{\mathbb{R}}^{-1}(C_\beta); \varphi_a \text{ convex on } \Sigma \text{ and } \varphi_a(\nu_i) \begin{cases} = -a_i & \text{for } i \notin I_\varnothing \\ \geq -a_i & \text{for } i \in I_\varnothing \end{cases} \right\}.$$

To show that this subset of $\mathbb{R}^r$ is a polyhedral cone, we will prove the following Proposition.

Proposition 3.14. Let $\Sigma \subset N_{\mathbb{R}}$ be a generalized fan with full-dimensional convex support, rational with respect to N. Then the vector space $\mathrm{SF}(\Sigma)$ of support-functions on Σ is canonically a linear subspace of $M_{\mathbb{R}}^{\Sigma(n)}$. The space $\mathrm{CSF}(\Sigma)$ of convex support-functions is a polyhedral cone in $\mathrm{SF}(\sigma)$, rational with respect to the lattice $M^{\Sigma(n)} \cap \mathrm{SF}(\Sigma) \subset M_{\mathbb{R}}^{\Sigma(n)}$. Furthermore, if there exists a strictly convex support function on Σ, then $\mathrm{CSF}(\Sigma)$ is full-dimensional in $\mathrm{SF}(\Sigma)$ and the (relative) interior of $\mathrm{CSF}(\Sigma)$ is the space of strictly convex support functions on Σ.

Proof. It is clear that the map

$$\mathrm{SF}(\Sigma) \to M_{\mathbb{R}}^{\Sigma(n)}, \varphi \mapsto (m_\sigma)_{\sigma \in \Sigma(n)}, \text{ where } \varphi|_\sigma = \langle m_\sigma, \cdot \rangle,$$

is a linear monomorphism, showing the first claim. Alternatively, one could try to descibe the space $\mathrm{SF}(\Sigma)$ by giving "compatibility equations", i.e. requiring that the

restrictions of $\langle m_{\sigma_i}, \cdot \rangle$ to $\sigma_i \cap \sigma_j$ agree for $\sigma_i, \sigma_j \in \Sigma(n)$.
For proving that $\mathrm{CSF}(\Sigma)$ is a (rational) cone, we will provide linear inequalities defining it. For every maximal cone $\sigma \in \Sigma(n)$ choose a lattice point $n_\sigma \in \mathrm{Int}(\sigma) \cap N$ contained in its interior. Then we claim that the set $\mathrm{CSF}(\Sigma) \subset \mathrm{SF}(\Sigma)$ is cut out by the (rational) equations

$$\langle m_\sigma, n_\sigma \rangle \leq \langle m_{\sigma'}, n_\sigma \rangle \text{ for } \sigma, \sigma' \in \Sigma(n), \sigma \neq \sigma'. \tag{3.14}$$

This follows more or less directly from a slightly modified version of Lemma 6.1.5 in [CLS11], but we still try to give an intuition, why this is true. If a point $(m_\sigma)_{\sigma \in \Sigma(n)}$ is contained in $\mathrm{CSF}(\Sigma)$, above equations will certainly hold, as a convex support function is always the minimum of the linear functions defined on its maximal cones. For the converse argument, assume you have a support function φ corresponding to a tuple $(m_\sigma)_{\sigma \in \Sigma(n)}$ satisfying (3.14). You will first show, that for any wall $\tau = \sigma \cap \sigma'$ with $\sigma, \sigma' \in \Sigma(n)$ equation (3.14) already implies $\langle m_\sigma, n \rangle \leq \langle m_{\sigma'}, n \rangle$ for all $n \in N_\mathbb{R}$ on the same side of (the hyperplane spanned by) τ as σ. Then the fact that φ is the minimum of the functions $\langle m_\sigma, \cdot \rangle$, and hence convex, follows quickly by the fact that one can connect any two points in the interior of maximal cones in Σ by a straight line "crossing" a finite number of walls.
Now assume there exists a strictly convex support function φ_0 on Σ defined by a tuple $(m_\sigma)_{\sigma \in \Sigma(n)}$. By definition, all inequalities in (3.14) are strict for φ_0. But then by general theory, the cone is full-dimensional (basically because "small" perturbations of φ_0 by support functions also satisfy the inequalities above) and its relative interior is defined by the strict versions of the inequalities (3.14). We have already seen that the strictly convex support functions satisfy these strict inequalities and arguments similar to above (or see Lemma 6.1.13 in [CLS11]) show the converse implication. □

Remark 3.15. Note that for the last part we do have to assume that there exists a strictly convex support function. Example 6.1.17 in [CLS11] gives a fan Σ without this property. As the linear functions on Σ are always convex, the empty set can then hardly be the relative interior of $\mathrm{CSF}(\Sigma)$.

Corollary 3.16 (see Proposition 14.4.3 in [CLS11])**.** For all admissible pairs $(\Sigma, I_\varnothing)$, the set $\tilde{\Gamma}_{\Sigma, I_\varnothing} \subset \mathbb{R}^r$ is a rational polyhedral cone with minimal face $\ker(\gamma_\mathbb{R})$ and relative interior $\tilde{\Gamma}^0_{\Sigma, I_\varnothing}$.

Proof. First we observe, that for any $a \in \tilde{\Gamma}_{\Sigma, I_\varnothing}$ the function φ_a is the unique convex support function on Σ satisfying

$$\varphi_a(\nu_i) \begin{cases} = -a_i & \text{for } i \notin I_\varnothing \\ \geq -a_i & \text{for } i \in I_\varnothing \end{cases}.$$

This is because, by Theorem 3.7, we have $\sigma = \mathrm{Cone}(\nu_i; \nu_i \in \sigma, i \notin I_\varnothing)$. Hence knowing a convex support function φ on ν_i for $i \notin I_\varnothing$ determines it on all maximal cones of Σ and hence on all of $|\Sigma|$. Thus an equivalent definition of $\tilde{\Gamma}_{\Sigma, I_\varnothing}$ (which was given in [CLS11]) is

$$\tilde{\Gamma}_{\Sigma,I_\varnothing} = \left\{ a \in \gamma_{\mathbb{R}}^{-1}(C_\beta);\ \text{there is } \varphi \in \mathrm{CSF}(\Sigma) \text{ with } \varphi(\nu_i) \begin{cases} = -a_i & \text{for } i \notin I_\varnothing \\ \geq -a_i & \text{for } i \in I_\varnothing \end{cases} \right\}.$$

From this it is clear, that we can write $\tilde{\Gamma}_{\Sigma,I_\varnothing}$ as follows: let $\Gamma \subset \mathbb{R}^r$ be the image of the rational polyhedral cone $\mathrm{CSF}(\Sigma) \subset \mathrm{SF}(\Sigma)$ under the evaluation map

$$\mathrm{ev}_{\boldsymbol{\nu}} : \mathrm{SF}(\Sigma) \to \mathbb{R}^r, \varphi \mapsto (\varphi(\nu_i))_{i=1}^r,$$

which respects the corresponding lattices $M^{\Sigma(n)} \cap \mathrm{SF}(\Sigma)$ and $\mathbb{Z}^r$. Then we have

$$\tilde{\Gamma}_{\Sigma,I_\varnothing} = \Gamma + \sum_{i \in I_\varnothing} \mathbb{R}_{\geq 0} e_i,$$

where $e_i \in \mathbb{R}^r$ is the ith standard basis vector. Hence $\tilde{\Gamma}_{\Sigma,I_\varnothing}$ is a polyhedral cone and it is rational as the sum of rational cones.

As for the minimal face of $\tilde{\Gamma}_{\Sigma,I_\varnothing}$ observe that if $a \in \tilde{\Gamma}_{\Sigma,I_\varnothing} \cap (-\tilde{\Gamma}_{\Sigma,I_\varnothing})$ then $\varphi_a, \varphi_{-a} \in \mathrm{CSF}(\Sigma)$. We claim that $\varphi_{-a} = -\varphi_a$. Indeed this follows as for $a' \in \tilde{\Gamma}_{\Sigma,I_\varnothing}$ the function $\varphi_{a'}$ is the unique function linear on all cones of Σ with $\varphi_{a'}(\nu_i) = a'_i$ for $i \notin I_\varnothing$.

Then φ_a and $-\varphi_a$ are convex functions, which easily implies that φ_a is linear, say $\varphi_a(u) = \langle m, u \rangle$ for some $m \in M_{\mathbb{R}}$. But as $\varphi_a(\nu_i) \geq -a_i$ and $-\varphi_a(\nu_i) \geq -(-a_i)$ for all i we have $\langle m, \nu_i \rangle = -a_i$, so indeed $-a \in \delta_{\mathbb{R}}(M_{\mathbb{R}})$. On the other hand all such a are contained in $\tilde{\Gamma}_{\Sigma,I_\varnothing} \cap (-\tilde{\Gamma}_{\Sigma,I_\varnothing})$.

For the claim about the relative interior observe, that for finite-dimensional real vector spaces V, W, a linear map $f : V \to W$ and polyhedral cones C, C' in V we have

$$\mathrm{RelInt}(f(C)) = f(\mathrm{RelInt}(C)) \text{ and } \mathrm{RelInt}(C + C') = \mathrm{RelInt}(C) + \mathrm{RelInt}(C').$$

This follows easily from the fact that the relative interior of the cone generated by a finite set S is the set of positive linear combinations of elements in S. Using this we see

$$\mathrm{RelInt}(\tilde{\Gamma}_{\Sigma,I_\varnothing}) = \mathrm{RelInt}(\Gamma) + \sum_{i \in I_\varnothing} \mathrm{RelInt}(\mathbb{R}_{\geq 0} e_i) = \mathrm{ev}_{\boldsymbol{\nu}}(\mathrm{RelInt}(\mathrm{CSF}(\Sigma))) + \sum_{i \in I_\varnothing} \mathbb{R}_{>0} e_i.$$

But as seen in Proposition 3.14, the relative interior of $\mathrm{CSF}(\Sigma)$ is the space of strictly convex support functions on Σ. Translating above description we obtain the definition of $\tilde{\Gamma}^0_{\Sigma,I_\varnothing}$. □

For the rest of this section, we will state several properties of the cones $\tilde{\Gamma}_{\Sigma,I_\varnothing}$, which allow us to work with them and to define the secondary fan. We will not give proofs but cite the corresponding statements from [CLS11].

We have now seen that the sets $\tilde{\Gamma}_{\Sigma,I_\varnothing} \subset \mathbb{R}^r$ are rational polyhedral cones with pairwise disjoint relative interior, which cover the cone $\gamma_{\mathbb{R}}^{-1}(C_\beta)$. One could hope that the set formed by these cones then defines a generalized fan $\tilde{\Sigma}_{\mathrm{GKZ}}$, and indeed, this is the

case. More precisely we have that every face of $\tilde{\Gamma}_{\Sigma,I_\varnothing}$ is again of the form $\tilde{\Gamma}_{\Sigma',I'_\varnothing}$ and for admissible pairs $(\Sigma, I_\varnothing), (\Sigma', I'_\varnothing)$ we have $\tilde{\Gamma}_{\Sigma',I'_\varnothing} \preceq \tilde{\Gamma}_{\Sigma,I_\varnothing}$ if and only if Σ refines Σ' and $I'_\varnothing \subset I_\varnothing$ (Lemma 14.4.6, [CLS11]). From this it already follows that $\tilde{\Sigma}_{\mathrm{GKZ}}$ is a generalized fan.
Observe that the minimal cone in $\tilde{\Sigma}_{\mathrm{GKZ}}$ is exactly $\ker(\gamma_\mathbb{R}) = \tilde{\Gamma}_{\{C_\nu\},\{1,\ldots,n\}}$. Thus if we quotient by this subspace, we obtain a proper fan. We define the **secondary fan** Σ_{GKZ} to be the collection of cones

$$\Gamma_{\Sigma,I_\varnothing} = \tilde{\Gamma}_{\Sigma,I_\varnothing}/\ker(\gamma_\mathbb{R}) \subset \mathbb{R}^r/\ker(\gamma_\mathbb{R}) = \widehat{G}_\mathbb{R}$$

for $\tilde{\Gamma}_{\Sigma,I_\varnothing} \in \tilde{\Sigma}_{\mathrm{GKZ}}$. One sees that it is a fan in $\widehat{G}_\mathbb{R}$ with support C_β. We still have that every face of a cone $\Gamma_{\Sigma,I_\varnothing}$ is of the form $\Gamma_{\Sigma',I'_\varnothing}$ and for admissible pairs $(\Sigma, I_\varnothing), (\Sigma', I'_\varnothing)$ we have $\Gamma_{\Sigma',I'_\varnothing} \preceq \Gamma_{\Sigma,I_\varnothing}$ if and only if Σ refines Σ' and $I'_\varnothing \subset I_\varnothing$. Finally the secondary fan gives the desired decomposition of the space of characters $\chi \in \widehat{G}$ according to their quotients: we have $\mathbb{C}^r /\!\!/_\chi G \cong X_\Sigma$ if $\chi \otimes 1 \in \mathrm{RelInt}(\Gamma_{\Sigma,I_\varnothing})$.

3.4.3. Computing the secondary fan

In the following section we will see how to actually compute the secondary fan from the data $\boldsymbol{\nu}, \boldsymbol{\beta}$. It turns out that the construction is very easy and geometric, but we will need some additional theory to prove that it gives the secondary fan.
The key idea for this proof is to identify the fan Σ_{GKZ} by finding the points in $\widehat{G}_\mathbb{R}$ which are contained in the interior of the maximal cones in Σ_{GKZ}. The connected components of this set will be the interiors of the maximal cones and thus their closures are the maximal cones, so we recover the original fan. As the secondary fan is rational, it will even suffice to decide for all points $\chi \otimes 1 \in C_\beta \subset \widehat{G}_\mathbb{R}$, $\chi \in \widehat{G}$, whether they are contained in the interior of such a maximal cone. The following statements are mostly taken from Theorem 14.3.14, Proposition 14.4.9 and Exercise 14.4.4. However, we take a different approach in some of the proofs and give an explicit construction of the secondary fan as the coarsest common refinement of all triangulations of $\boldsymbol{\beta}$.

Generic characters

A character $\chi \in \widehat{G}$ is called **generic** if $\chi \otimes 1 \in C_\beta$ and for all $\boldsymbol{\beta}' \subset \boldsymbol{\beta}$ with $\dim \mathrm{Cone}(\boldsymbol{\beta}') < \dim \widehat{G}_\mathbb{R}$ we have $\chi \otimes 1 \notin \mathrm{Cone}(\boldsymbol{\beta}')$. We cite the following theorem from [CLS11].

Theorem (see Theorem 14.3.14 in [CLS11])**.** Let $\chi \in \widehat{G}$ be a character with $\chi \otimes 1 \in C_\beta$. Then the following are equivalent:

(a) χ is generic.

(b) Every vertex of P_χ has precisely $\dim G$ nonzero coordinates.

(c) P_χ is simple of dimension $r - \dim G$, every virtual facet $F_{i,\chi} \subset P_\chi$ is either empty or a genuine (codimension 1) facet and $F_{i,\chi} \neq F_{j,\chi}$ if $i \neq j$ and both $F_{i,\chi}, F_{j,\chi}$ are nonempty.

(d) $(\mathbb{C}^r)^s_\chi = (\mathbb{C}^r)^{ss}_\chi$.

We now use this theorem to prove the following result (Proposition 14.4.9. in [CLS11]). Our proof will be longer than the proof given in [CLS11], but it will give a first intuition how the value of a affects $\Sigma(a), I_\varnothing(a)$ when varying a slightly.

Proposition 3.17. Let $\chi = \chi^a \in \widehat{G}$ be a character with $\chi \otimes 1 \in C_\beta$. Let $(\Sigma, I_\varnothing)$ be the unique admissible pair with $\chi \otimes 1 \in \mathrm{RelInt}(\Gamma_{\Sigma, I_\varnothing})$. Then the following are equivalent

(a) χ is generic.

(b) Σ is simplicial and $i \mapsto \mathrm{Cone}(\nu_i)$ induces a bijection $\{1, \dots, r\} \setminus I_\varnothing \cong \Sigma(1)$.

(c) $\Gamma_{\Sigma, I_\varnothing}$ is a chamber of the secondary fan.

Proof. For the equivalence of (a) and (b) here, we use the previous theorem. More precisely, we note that the statement (c) about the polyhedron P_χ in Theorem 14.3.14 is equivalent to statement (b) here. Note that as $\chi \otimes 1 \in \mathrm{RelInt}(\Gamma_{\Sigma, I_\varnothing})$, we have that Σ is the normal fan of P_a. One then checks that P_χ being simple of dimension $r - \dim G$ means P_a is simple of the same dimension (and hence full-dimensional in $M_\mathbb{R}$). This translates to Σ being simplicial. Furthermore, the nonempty genuine facets correspond to the rays of Σ and at the same time, the facet $F_{i,\chi}$ is nonempty iff $i \notin I_\varnothing$. This shows the equivalence (a) $\iff$ (b).
Now note that statements (b) and (c) make sense without referring directly to a specific character χ, and in fact they are equivalent as statements about admissible pairs $(\Sigma, I_\varnothing)$. For (c) $\implies$ (b) assume that the cone $\Gamma_{\Sigma, I_\varnothing}$ is a chamber, i.e. a maximal cone in Σ_{GKZ}. This is equivalent to saying that there is no pair $(\Sigma', I'_\varnothing) \neq (\Sigma, I_\varnothing)$ with $\Gamma_{\Sigma, I_\varnothing} \preceq \Gamma_{\Sigma', I'_\varnothing}$. But as we saw, such a face relation is satisfied iff Σ' refines Σ and $I_\varnothing \subset I'_\varnothing$. The fact that $\Gamma_{\Sigma, I_\varnothing}$ is full-dimensional and $\chi^a \otimes 1 \in \mathrm{Int}(\Gamma_{\Sigma, I_\varnothing})$ implies that we can change a by a small amount without affecting $\Sigma(a), I_\varnothing(a)$.
First assume that Σ is not simplicial and take a maximal cone $\sigma \in \Sigma$ not being simplicial. The cone σ is spanned by vectors ν_j for $j \in J \subset \{1, \dots, n\} \setminus I_\varnothing$, where $J = \{j \notin I_\varnothing ; \nu_j \in \sigma\}$. As σ is not simplicial, we have $|J| > n$. Choose $J' \subset J$ such that $(\nu_l)_{l \in J'}$ forms a basis of $N_\mathbb{R}$, take some $j \in J \setminus J'$ and consider $a' = a + \epsilon e_j$ for $\epsilon > 0$ small. By assumption $I_\varnothing(a') = I_\varnothing$ and $J \subset \{1, \dots, n\} \setminus I_\varnothing(a')$, so $\varphi_{a'}(\nu_k) = -a'_k$ for $k \in J$. At the same time, $\varphi_{a'}$ remains linear on σ. This gives a contradiction, as the values on the remaining vectors ν_k, $k \in J \setminus \{j\}$, which span $N_\mathbb{R}$, are unchanged. Hence Σ is simplicial and thus a proper fan (i.e. its cones are strongly convex and hence spanned by their rays).
For the second part observe that $I_\varnothing$ is maximal with $(\Sigma, I_\varnothing)$ being admissible. Now by Theorem 3.7 we have that for each ray $\rho \in \Sigma(1)$ there is an index $i \notin I_\varnothing$ with $\rho = \mathrm{Cone}(\nu_i)$. If two distinct indices $i, j \notin I_\varnothing$ satisfy this, then one sees from the definition of φ_a that increasing a_i slightly does not change φ_a and one obtains $i \in I_\varnothing(a + \epsilon e_i)$, a contradiction. Hence the map $\Sigma(1) \to \{1, \dots, r\} \backslash I_\varnothing, \rho \mapsto i$ is well-defined and it is necessarily injective. To show surjectivity assume that there is $i \notin I_\varnothing$ with $\mathrm{Cone}(\nu_i) \notin \Sigma(1)$. Then necessarily there exists some cone $\sigma \in \Sigma$ with $\nu_i \in \mathrm{RelInt}(\sigma)$. Again we increase a_i slightly and see that φ_a does not change on σ, as its values on the rays of σ (spanned

by ν_j with $i \neq j \notin I_\varnothing$) are unchanged. Hence as above $I_\varnothing$ increases, giving a contradiction.
For (b) $\implies$ (c) assume we have $(\Sigma, I_\varnothing)$ with Σ simplicial and a bijection $\{1,\dots,r\} \smallsetminus I_\varnothing \cong \Sigma(1)$. As seen above, if $\Gamma_{\Sigma,I_\varnothing}$ is not maximal, we find $(\Sigma', I'_\varnothing)$ corresponding to a chamber $\Gamma_{\Sigma',I'_\varnothing}$ with Σ' refining Σ and $I_\varnothing \subset I'_\varnothing$. By applying the same arguments as in (c) $\implies$ (b), we may assume that Σ' is simplicial and that there is a bijection $\{1,\dots,n\} \smallsetminus I'_\varnothing \cong \Sigma'(1)$. Hence $\Sigma'(1) \subset \Sigma(1)$.
Now for any maximal cone $\sigma' \in \Sigma'(n)$ there exists a cone $\sigma \in \Sigma$ with $\sigma' \subset \sigma$. It follows from the fact that Σ is a fan that $\sigma'(1) \subset \sigma(1)$ and as σ is simplicial and full-dimensional, we have equality and thus $\sigma' = \sigma$. Hence $\Sigma'(n) \subset \Sigma(n)$ and as both fans have the same support, we have $\Sigma = \Sigma'$ and consequently $I_\varnothing = I'_\varnothing$. Thus $\Gamma_{\Sigma,I_\varnothing}$ is a chamber. □

Note that we proved the equivalence $(b) \iff (c)$ without referring to Theorem 14.3.14 in [CLS11].

Remark 3.18. A fan Σ is called a **triangulation** of a vector configuration $V = \{v_1,\dots,v_r\} \subset \mathbb{R}^d$ if it is simplicial with support $\mathrm{Cone}(V)$ and $\Sigma(1) \subset \{\mathrm{Cone}(v_i); i = 1,\dots,r\}$. Note that a triangulation, for us, is not a dissection of the convex hull $\mathrm{Conv}(V)$ of V into simplices. A short argument shows that any set $V \subset \mathbb{R}^d$ can only have finitely many triangulations, with $2^{\binom{r}{d}}$ being an upper bound on this number.
For a vector configuration $\boldsymbol{\nu}$ in $N_\mathbb{R}$ and a fan Σ in $N_\mathbb{R}$ we say that Σ is a **regular fan on $\boldsymbol{\nu}$** if $\Sigma = \Sigma(a)$ for some $a \in \gamma_\mathbb{R}^{-1}(C_\beta)$. If in addition Σ is simplicial we say it is a **regular triangulation of $\boldsymbol{\nu}$**. It is clear that the map sending $\Gamma_{\Sigma,I_\varnothing}$ to Σ is a surjection from Σ_{GKZ} to the set of regular fans on $\boldsymbol{\nu}$. Proposition 3.17 then shows that this map restricts to a bijection from the chambers of Σ_{GKZ} to the set of regular triangulations of $\boldsymbol{\nu}$.

The coarsest common refinement

By the Proposition above the fan Σ_{GKZ} is directly determined by the data $\boldsymbol{\beta}$ as described at the beginning of the section. However, taking the closures of the connected components of

$$C_\beta \smallsetminus \bigcup(\mathrm{Cone}(\boldsymbol{\beta}'); \boldsymbol{\beta}' \subset \boldsymbol{\beta}, \dim\, \mathrm{Cone}(\boldsymbol{\beta}') < \dim\, C_\beta)$$

and identifying them as polyhedral cones seems an overly complicated procedure. Therefore we will present a construction, which is more algorithmically trackable. It will have big overlaps with Proposition 15.2.1. and Lemma 15.2.10 from [CLS11], but it will be applicable for a more general situation. Many ideas presented here were also inspired by [DLRS10], especially by Remark 5.4.8..
Let $\boldsymbol{\beta} = \{\beta_1,\dots,\beta_r\} \subset \widehat{G}_\mathbb{R}$ be a finite set of vectors in $\widehat{G}_\mathbb{R}$ and let $C_\beta \subset \widehat{G}_\mathbb{R}$ be the cone spanned by them. A subset $J \subset \{1,\dots,r\}$ is called a **$\boldsymbol{\beta}$-basis** if $(\beta_j)_{j\in J}$ is a basis of $\widehat{G}_\mathbb{R}$. Let $\mathcal{J} \subset \mathcal{P}(\{1,\dots,r\})$ denote the set of $\boldsymbol{\beta}$-bases. In the following you should

think of the elements J of $\mathcal{J}$ not as mere sets of indices, but as the simplicial cones

$$\mathrm{Cone}(\boldsymbol{\beta}_J) = \mathrm{Cone}(\beta_j; j \in J) \subset \widehat{G}_{\mathbb{R}}$$

spanned by them. Note that the formula above also makes sense if J is not a $\boldsymbol{\beta}$-basis.

For two fans Σ, Σ' in $\widehat{G}_{\mathbb{R}}$ with common support $|\Sigma| = |\Sigma'|$ we define the **coarsest common refinement of Σ and Σ'** as

$$\Sigma \wedge \Sigma' = \{\sigma \cap \sigma'; \sigma \in \Sigma, \sigma' \in \Sigma'\}, \tag{3.15}$$

the set of intersections of cones in Σ with cones in Σ'. Then we have the following result.

Proposition 3.19. The set $\Sigma \wedge \Sigma'$ is a fan in $\widehat{G}_{\mathbb{R}}$ with support $|\Sigma|$. It is the coarsest fan with this support refining both Σ and Σ', that is every other fan with this property refines $\Sigma \wedge \Sigma'$.

Proof. It is clear that all elements of $\Sigma \wedge \Sigma'$ are strongly convex polyhedral cones, as the elements of Σ were already strongly convex. Let $\sigma \in \Sigma$, $\sigma' \in \Sigma'$ and $\tau_0 \leq \sigma \cap \sigma'$. We have to show that τ_0 is again the intersection of a cone in Σ with a cone in Σ'. By Corollary 2.3 there are faces $\tau \leq \sigma, \tau' \leq \sigma'$ with $\tau \cap \tau' = \tau_0$. But as Σ, Σ' are fans, we have $\tau \in \Sigma, \tau' \in \Sigma'$.
Let now $\sigma_1, \sigma_2 \in \Sigma$ and $\tau_1, \tau_2 \in \Sigma'$, then we have to show $(\sigma_1 \cap \tau_1) \cap (\sigma_2 \cap \tau_2) \leq \sigma_1 \cap \tau_1$. But note that $\sigma_1 \cap \sigma_2 \leq \sigma_1$ and $\tau_1 \cap \tau_2 \leq \tau_1$. From this it follows easily that

$$(\sigma_1 \cap \tau_1) \cap (\sigma_2 \cap \tau_2) = (\sigma_1 \cap \sigma_2) \cap (\tau_1 \cap \tau_2) \leq \sigma_1 \cap \tau_1.$$

For every fan $\tilde{\Sigma}$ refining Σ and Σ' and a cone $\tilde{\sigma} \in \tilde{\Sigma}$ there is a cone $\sigma \in \Sigma$ with $\tilde{\sigma} \subset \sigma$ and a cone $\sigma' \in \Sigma'$ with $\tilde{\sigma} \subset \sigma'$, hence $\tilde{\sigma} \subset \sigma \cap \sigma' \in \Sigma \wedge \Sigma'$. This shows the characterization as coarsest common refinement of Σ, Σ'. □

We remark that we can now also define the coarsest common refinement of a finite number of fans $\Sigma_1, \ldots, \Sigma_k$ by iterating above process. The characterization as the coarsest fan refining all fans Σ_i shows that the result does not depend on the order in which we apply $\wedge$.
We will now look at the coarsest common refinement of all triangulations of the set $\boldsymbol{\beta}$. This will be a fan with support C_β and we will see that it is exactly the secondary fan. Below we will give another description of this fan, which allows us to compute it more efficiently.
For a set $A \subset \mathcal{J}$ we denote

$$C_A = \bigcap_{J \in A} \mathrm{Cone}(\boldsymbol{\beta}_J) \subset \widehat{G}_{\mathbb{R}}$$

the cone obtained as the intersection of the cones corresponding to $\boldsymbol{\beta}$-bases in A. We define the **coarsest common refinement of** β as the set

$$\operatorname{ccr}(\boldsymbol{\beta}) = \{\tau \leq C_A; A \subset \mathcal{J} \text{ maximal with } \dim C_A = \dim \widehat{G}_{\mathbb{R}}\} \tag{3.16}$$

of polyhedral cones in $\widehat{G}_{\mathbb{R}}$. Then we claim the following:

Theorem 3.20. The set $\operatorname{ccr}(\boldsymbol{\beta})$ is the coarsest common refinement of all triangulations of the set $\boldsymbol{\beta}$. In particular it is a fan in $\widehat{G}_{\mathbb{R}}$ with support C_β. Let

$$C = \bigcup(\operatorname{Cone}(\boldsymbol{\beta}_J); J \subset \{1, \ldots, r\}, \operatorname{codim} \operatorname{Cone}(\boldsymbol{\beta}_J) = 1).$$

Then $\operatorname{ccr}(\boldsymbol{\beta})$ is the unique fan Σ with support C_β characterized by the following equivalent properties:

(a) We have that the union of all codimension 1 cones in Σ is equal to C.

(b) We have that the union of all interiors of maximal cones in Σ is equal to $C_\beta \setminus C$.

In particular we have $\operatorname{ccr}(\boldsymbol{\beta}) = \Sigma_{\mathrm{GKZ}}$ for $\boldsymbol{\beta}$ the Gale-dual of $\boldsymbol{\nu}$ as above.

To prove the Theorem we will need to show that there are enough triangulations to have at least every simplicial cone spanned by vectors in $\boldsymbol{\beta}$ contained in one of them. This will follow from the Lemma below.

Lemma 3.21. Let $\boldsymbol{\beta} = \{\beta_1, \ldots, \beta_r\} \subset \widehat{G}_{\mathbb{R}}$ be a finite set of vectors spanning $\widehat{G}_{\mathbb{R}}$ as a vector space and Σ a triangulation of $\boldsymbol{\beta}$. Let $\beta_{r+1} \in \widehat{G}_{\mathbb{R}}$ be an additional vector. Then there exists a triangulation Σ' of $\boldsymbol{\beta} \cup \{\beta_{r+1}\}$ whose restriction to $C_\beta = \operatorname{Cone}(\boldsymbol{\beta})$ is Σ.

Proof. If $\beta_{r+1} \in C_\beta$ we take $\Sigma' = \Sigma$. Otherwise look at the set $\mathcal{F}$ of facets F of C_β, such that β_{r+1} is strictly on the opposite side of (the hyperplane spanned by) F from C_β. As $\beta_{r+1} \notin C_\beta$ we have that $\mathcal{F}$ is nonempty. All facets $F \in \mathcal{F}$ are simplicial and hence the cones $C_F = \operatorname{Cone}(F, \beta_{r+1})$ are simplicial too. We define Σ' as the union of Σ with all cones $C_F, F \in \mathcal{F}$ and their facets. Clearly this is a collection of strongly convex simplicial cones. One easily checks that its support is exactly $\operatorname{Cone}(\boldsymbol{\beta} \cup \{\beta_{r+1}\})$. It is clear that faces of cones in Σ' are again in Σ'. For the fact that intersections of cones $\sigma_1, \sigma_2 \in \Sigma'$ are again in Σ' one can restrict oneself to intersections, where $\sigma_1 = C_F$ for some $F \in \mathcal{F}$. If $\sigma_2 \in \Sigma$, one uses that $C_F \cap C_\beta = F$ and sees $\sigma_1 \cap \sigma_2 = F \cap \sigma_2$, which is a face of σ_1 as this cone is simplicial and a face of σ_2 as Σ was a fan. For $\sigma_2 = C_{F'}, F' \in \mathcal{F}$ one easily sees $\sigma_1 \cap \sigma_2 = \operatorname{Cone}(F \cap F', \beta_{r+1})$ and as $F \cap F' \leq F$ one obtains $\sigma_1 \cap \sigma_2 \leq \sigma_1$. This concludes the proof. □

We are now ready to prove the theorem.

Proof of Theorem 3.20. The crucial idea to show that $\operatorname{ccr}(\boldsymbol{\beta})$ is the coarsest common refinement of all triangulations of $\boldsymbol{\beta}$ is to see that for every $\boldsymbol{\beta}$-basis J we can find a triangulation Σ_J with $\operatorname{Cone}(\boldsymbol{\beta}_J) \in \Sigma_J$. Indeed, the cone $\operatorname{Cone}(\boldsymbol{\beta}_J)$ is simplicial, so it is a triangulation of $\{\beta_j; j \in J\}$. Then we construct Σ_J by successively applying Lemma 3.21 and adding the vectors $\beta_i, i \notin J$ one at a time. This shows, that the set of maximal cones occuring in some triangulation of $\boldsymbol{\beta}$ is exactly $\{\operatorname{Cone}(\boldsymbol{\beta}_J); J \in \mathcal{J}\}$.

One then easily sees that the cones C_A with $A \subset \mathcal{J}$ maximal with dim C_A = dim $\widehat{G}_{\mathbb{R}}$ are exactly the maximal cones in the coarsest common refinement of all triangulations. Hence this fan agrees with $\mathrm{ccr}(\boldsymbol{\beta})$.
As the interior of a full-dimensional cone σ is the complement (in σ) of its codimension 1 faces, one sees immediately that (a) is equivalent to (b). We have seen before, that condition (b) uniquely determines the secondary fan Σ_{GKZ}. We will now show that $\mathrm{ccr}(\boldsymbol{\beta})$ satisfies condition (b).
It is clear that the cones C_A for $A \in \mathcal{A} = \{A \subset \mathcal{J}$ maximal with dim C_A = dim $M_{\mathbb{R}}\}$ are exactly the full-dimensional cones in $\mathrm{ccr}(\boldsymbol{\beta})$. For two full-dimensional fans Σ, Σ' with $|\Sigma| = |\Sigma'|$ we have that the interiors of the maximal cones of $\Sigma \wedge \Sigma'$ are exactly

$$\bigcup_{\sigma\in\Sigma(n),\sigma'\in\Sigma(n)} \mathrm{Int}(\sigma\cap\sigma') = \bigcup_{\sigma\in\Sigma(n),\sigma'\in\Sigma(n)} \mathrm{Int}(\sigma)\cap\mathrm{Int}(\sigma')$$
$$= \left(\bigcup_{\sigma\in\Sigma(n)} \mathrm{Int}(\sigma)\right) \cap \left(\bigcup_{\sigma'\in\Sigma'(n)} \mathrm{Int}(\sigma')\right).$$

A point $p \in |\Sigma|$ is contained in this union iff it is not contained in any codimension 1 cone of Σ or Σ'. Applied to the coarsest common refinement of all triangulations of $\boldsymbol{\beta}$ we claim that the codimension 1 cones appearing in some triangulation are exactly the cones $\mathrm{Cone}(\boldsymbol{\beta}_J)$ where $J \subset \{1,\dots,r\}, |J| = \dim\widehat{G}_{\mathbb{R}}-1$ with $(\beta_j)_{j\in J}$ linearly independent. Indeed, any such cone can be enriched to a simplicial full-dimensional cone by adding some β_i and such a cone can be enriched to a triangulation as above. By applying the theorem of Carathéodory (see Proposition 2.1), one sees that the union of above codimension 1 cones is exactly C as defined above. This concludes the proof. □

Remark 3.22. Assume we have an identification $\widehat{G} = \mathbb{Z}^d \subset \mathbb{R}^d = \widehat{G}_{\mathbb{R}}$ and we have $\boldsymbol{\beta} \subset \mathbb{Z}^d$. Then rational polyhedral cones in $\mathbb{R}^n$ can be described by finite sets of vectors in $\mathbb{Z}^d$ generating its rays. For us then computing a polyhedral cone will mean computing such a finite set of rational ray generators.
Going through the construction of the fan $\mathrm{ccr}(\boldsymbol{\beta})$, we see that the algorithmic operations we have to carry out are

- deciding for a finite set of vectors in $\mathbb{Z}^d$ if it spans the entire $\mathbb{R}^d$,
- computing the intersection of two rational polyhedral cones,
- computing the faces of a rational polyhedral cone.

Each of these operations must only be carried out a finite number of times and all of them can be carried out by a computer in finite time. For an implementation see for example `polymake` [GJ00]. Hence we have described, in principle, an algorithm for computing the secondary fan. However, this algorithm is far from being efficient. For an implementation of a more advanced algorithm using Gröbner fans see [Jen].

4. The geometry of the secondary fan

In the last chapter we have defined the secondary fan Σ_{GKZ} starting from Gale-dual vector configurations $\boldsymbol{\nu} \subset N_{\mathbb{R}}$ and $\boldsymbol{\beta} \subset \widehat{G}_{\mathbb{R}}$. We now want to understand how its geometry is reflected in the quotients associated to its cones.
More concretely, assume we have given two chambers $\Gamma_{\Sigma,I_{\varnothing}}, \Gamma_{\Sigma',I'_{\varnothing}}$ of the secondary fan, which share a common wall. This wall will be of the form $\Gamma_{\Sigma_0,I_{\varnothing,0}}$. We will see which pairs $(\Sigma, I_{\varnothing}), (\Sigma', I'_{\varnothing}), (\Sigma_0, I_{\varnothing,0})$ can occur. Moreover it will turn out that they are related by two different types of operations: star-subdivisions and flips. These operations arise on the level of fans, but also induce well-known algebraic-geometric transformations. Using a result from [DLRS10] we will be able to improve results from [CLS11] to obtain a more global way to tell, which of the above operations correspond to which walls. We will also use these results to derive statements about the graph of all regular triangulations of the vector configuration $\boldsymbol{\nu}$.

4.1. Star-subdivisions and flips

Star subdivisions and flips are transformations between triangulations of a given vector configuration, which are local in some sense. From a purely combinatorial point of view both are described by the same recipe and sometimes (as in [DLRS10]) they are both referred to as flips. Our account here is based on Lemma 2.4.2 and Theorem 4.4.1 in [DLRS10].
First we describe the local part of the flip operation. Assume we are given a set $v_1, \dots, v_s \in N_{\mathbb{R}}$ of s vectors which span an $(s-1)$-dimensional subspace space of $N_{\mathbb{R}}$. Such a configuration is called of **corank** 1. Then there is a nontrivial linear relation

$$\sum_{i \in J_+} \lambda_i v_i + \sum_{j \in J_-} \lambda_j v_j = 0,$$

unique up to multiplication by a nonzero scalar, where $\lambda_i > 0$ for $i \in J_+$ and $\lambda_j < 0$ for $j \in J_-$. We note that the roles of J_+ and J_- can be swapped by multiplying the relation by a negative number. This gives an oriented circuit C of the vector configuration $v_1, \dots, v_s$ where $\mathrm{C}_k = +$ for $k \in J_+$, $\mathrm{C}_k = -$ for $k \in J_-$ and $\mathrm{C}_k = 0$ for $k \in J_0 = \{1, \dots, s\} \smallsetminus (J_+ \cup J_-)$. For a subset $J \subset \{1, \dots, s\}$ let

$$C_J = \mathrm{Cone}(v_k; k \in J).$$

Then in Lemma 2.4.2 of [DLRS10] it is shown that the two unique triangulations of $v_1, \dots, v_s$ are

$$\Sigma_+^{\mathrm{C}} = \{C_J; J \subset \{1,\dots,s\}, J_+ \notin J\} \text{ and } \Sigma_-^{\mathrm{C}} = \{C_J; J \subset \{1,\dots,s\}, J_- \notin J\}.$$

Note that the maximal cones of the triangulations are

$$\Sigma_+^{\mathrm{C}}(s-1) = \{C_{\{1,\dots,s\}\setminus\{j\}}; j \in J_+\} \text{ and } \Sigma_-^{\mathrm{C}}(s-1) = \{C_{\{1,\dots,s\}\setminus\{j\}}; j \in J_-\}.$$

Now we come to the global description of a flip. Assume we have a vector configuration $\boldsymbol{\nu} = \{\nu_1, \dots, \nu_r\} \subset N_{\mathbb{R}}$ and two simplicial fans Σ_+, Σ_- in $N_{\mathbb{R}}$ that are triangulations of $\boldsymbol{\nu}$, i.e. their support is C_ν and their rays are contained in $\{\mathrm{Cone}(\nu_i); i = 1, \dots, r\}$. Let C be a circuit of $\boldsymbol{\nu}$ with positive indices J_+ and negative indices J_- and let $\Sigma_+^{\mathrm{C}}, \Sigma_-^{\mathrm{C}}$ be the triangulations of the vectors $\nu_k, k \in J_+ \cup J_-$ from above. We want to define what it means that Σ_+, Σ_- differ by a **flip induced by c**.
The first condition is that the fan Σ_+^{C} is contained in the fan Σ_+ and similarly $\Sigma_-^{\mathrm{C}} \subset \Sigma_-$. Now for every maximal simplex σ_0 of Σ_+^{C} or Σ_-^{C} and every maximal simplex $\sigma \in \Sigma_+(n)$ containing σ_0 we consider the cone spanned by the rays in $\sigma(1) \setminus \sigma_0(1)$. This is called the **link simplex** of σ_0 in σ. For σ_0 fixed as above, the union of all link simplices of σ_0 in maximal simplices $\sigma \in \Sigma_+(n)$ containing it, is called the **link** of σ_0 in Σ_+. As a second condition we require that for all maximal simplices σ_0 of Σ_+^{C} or Σ_-^{C} the link L in Σ_+ agrees. Note that this uniquely determines Σ_+, Σ_- on the union U of maximal cones of Σ_+ containing some cone of Σ_+^{C}, which is equal to the corresponding union of maximal cones of Σ_-^{C}.
Finally we require that "outside" of U the fans Σ_+ and Σ_- agree. That is, removing from Σ_+ all cones contained in U and adding the cones of Σ_- contained in U produces the fan Σ_-.
In case $J_+ = \{i\}, |J_-| \geq 2$ we say that Σ_- is the **star-subdivision** of Σ_+ at ν_i (or at $\mathrm{Cone}(\nu_i)$). One sees easily that this generalizes the definition of the star-subdivision of a smooth chamber $\sigma \in \Sigma$ for a simplicial fan Σ given in Section 2.3.3.

4.2. Wall-crossing

Now we return to the situation described at the beginning of the section. We have two chambers $\Gamma_{\Sigma, I_\varnothing}, \Gamma_{\Sigma', I'_\varnothing}$ of the secondary fan, which share a common wall $\tau = \Gamma_{\Sigma_0, I_{\varnothing,0}}$. By Theorem 3.20 we know that τ is contained in the union of all cones $\mathrm{Cone}(\boldsymbol{\beta}_J)$, where $J \subset \{1, \dots, r\}$ and $\mathrm{codim}\, \mathrm{Cone}(\boldsymbol{\beta}_J) = 1$. Note that this implies that the elements of $\boldsymbol{\beta}$ contained in the hyperplane spanned by τ already span this hyperplane. Otherwise all cones $\mathrm{Cone}(\boldsymbol{\beta}_J)$ above would intersect τ in codimension at least 1 and thus τ cannot be contained in their union. We conclude that the wall τ gives us a cocircuit C of the vector configuration $\boldsymbol{\beta}$ unique up to sign. Note that this cocircuit must satisfy $|J_+|, |J_-| \geq 1$. Otherwise all vectors of $\boldsymbol{\beta}$ lie in one of the closed half spaces defined by the hyperplane spanned by τ. This is a contradiction to τ being a wall between two chambers of Σ_{GKZ}. Choose the orientation of C in such a way, that the vectors of $\boldsymbol{\beta}$

lying on the same side of τ as $\Gamma_{\Sigma,I_\varnothing}$ are positive. By Gale-duality we obtain a circuit C of the vector configuration $\boldsymbol{\nu}$ (see Section 2.5.3).

Theorem 4.1. Let $\Gamma_{\Sigma,I_\varnothing}, \Gamma_{\Sigma',I'_\varnothing}$ as above be two chambers of the secondary fan Σ_{GKZ} sharing a common wall $\tau = \Gamma_{\Sigma_0,I_{\varnothing,0}}$. Let C be the circuit of $\boldsymbol{\nu}$ induced by τ, oriented as described above. Then C induces a flip between Σ and Σ', where $\Sigma = \Sigma_+^{\mathrm{C}}, \Sigma' = \Sigma_-^{\mathrm{C}}$. Moreover, Σ_0 is the finest fan refined by both Σ and Σ'. As τ is a wall, we have $|J_+|, |J_-| \geq 1$. If the circuit C satisfies $|J_+| \geq 2, |J_-| \geq 2$, we have $I_\varnothing = I'_\varnothing = I_{\varnothing,0}$. If $J_+ = \{i\}, |J_-| \geq 2$ we have $i \in I_\varnothing$ and $I'_\varnothing = I_{\varnothing,0} = I_\varnothing \setminus \{i\}$. The corresponding statement for the roles of J_+, J_- reversed holds as well.

Proof. The first part of the theorem is exactly the content of Theorem 5.4.12. in [DLRS10]. The second part is also mentioned in [DLRS10] but it is explicitly given in Theorem 15.3.6., Lemma 15.3.7. and Theorem 15.3.13. of [CLS11]. □

Following [CLS11] we say that a wall τ is a **divisorial wall** if its oriented circuit satisfies $|J_+| = 1, |J_-| \geq 2$ (or vice versa) and a **flipping wall** if it satsifies $|J_+|, |J_-| \geq 2$. Note that in the last Theorem, the case $|J_+| = 1, |J_-| = 1$ is not treated. If such a circuit of $\boldsymbol{\tau}$ exists, we have indices i, j such that $\lambda_i \nu_i - \lambda_j \nu_j = 0$ for $\lambda_i, \lambda_j > 0$. This implies that ν_i, ν_j generate the same ray in $N_\mathbb{R}$ and this can create somewhat pathological behaviour. An example for the type of wall-crossing that can occur in such a case is treated in Section 5.2.1.
To avoid such behaviour, we say that a vector configuration $\boldsymbol{\nu}$ is **geometric** if all vectors ν_i are nonzero and the rays $\mathrm{Cone}(\nu_i)$ are pairwise distinct.
In Section 2.3.3 we have seen that star-subdivision of the fan corresponded to blowing up a torus fixed point. In general if Σ^* is the star-subdivision of Σ at ν_i it is shown in Theorem 15.3.6 of [CLS11] that the morphism $X_{\Sigma^*} \to X_\Sigma$ induced by the fact that Σ^* refines Σ is a birational map with exceptional locus of codimension 1. In fact the exceptional locus is the divisor D_i in X_{Σ^*} corresponding to $\mathrm{Cone}(\nu_i) \in \Sigma^*(1)$.
On the other hand we have seen that for flipping walls $\Gamma_{\Sigma_0,I_{\varnothing,0}}$ separating $\Gamma_{\Sigma,I_\varnothing}$ and $\Gamma_{\Sigma',I'_\varnothing}$ the rays of Σ, Σ' and Σ_0 agree and Σ, Σ' refine Σ_0. By Theorem 15.3.13 in [CLS11] this leads to surjective toric morphisms $\phi : X_\Sigma \to X_{\Sigma_0}$ and $\phi' : X_{\Sigma'} \to X_{\Sigma_0}$ that are birational with exceptional locus of codimension ≥ 2. Then the birational map

$$\phi'^{-1} \circ \phi : X_\Sigma \dashrightarrow X_{\Sigma'}$$

is a flip (in the sense of Section 2.1.2). In Section 15.4 of [CLS11] the link from these flips to the minimal model program is established. As we will not use any of these results in the further sections and as we have nothing new to add, we do not go into further detail here.

4.3. Applications

We now want to present some results inspired by Theorem 4.1 concerning the geometry of the secondary fan. We define the **graph G_ν of regular triangulations** of a vector configuration $\boldsymbol{\nu} = \{\nu_1, \dots, \nu_r\} \subset N \subset N_\mathbb{R}$. Its vertices are the chambers of the secondary fan Σ_{GKZ} and two chambers are connected by an edge if they share a common wall τ. We immediately have the following result.

Proposition 4.2. The graph G_ν is connected.

Proof. Take two chambers $\Gamma_{\Sigma,I_\varnothing}, \Gamma_{\Sigma',I'_\varnothing}$ of the secondary fan and choose two points b, b' in their interior sufficiently general such that the line segment connecting them only meets walls of the secondary fan (and not cones with codimension strictly higher than 1). Here we use that the support of the secondary fan is the convex cone C_β. Then the sequence of chambers which the line segment meets defines a path in G_ν connecting $\Gamma_{\Sigma,I_\varnothing}$ and $\Gamma_{\Sigma',I'_\varnothing}$. □

We can now give some results concerning the structure of this graph. For a vector configuration $\boldsymbol{\beta} = \{\beta_1, \dots, \beta_r\} \subset \widehat{G}_{\mathbb{R}}$ we say that an element $\beta_i \in \boldsymbol{\beta}$ is in **extremal position** if

$$\beta_i \notin \mathrm{Cone}(\boldsymbol{\beta}_{\{1,\dots,r\}\setminus\{i\}}) = \mathrm{Cone}(\beta_1, \dots, \widehat{\beta}_i, \dots, \beta_r).$$

Proposition 4.3. For a cone $\Gamma_{\Sigma,I_\varnothing}$ of the secondary fan we have

$$I_\varnothing = \{i; \beta_i \text{ is in extremal position and } \beta_i \in \Gamma_{\Sigma,I_\varnothing}\} \tag{4.1}$$

Moreover, for β_i in extremal position we have that $(N_{\mathbb{R}}, \{i\})$ is an admissible pair with $\Gamma_{N_{\mathbb{R}},\{i\}} = \mathrm{Cone}(\beta_i)$.

Proof. Let $i \in \{1, \dots, r\}$, then Lemma 2.4 implies that β_i is in extremal position iff $\nu_i \in \mathrm{Cone}(\nu_1, \dots, \widehat{\nu}_i, \dots, \nu_r)$. If this is the case then one sees easily that for $a = e_i \in \mathbb{R}^r$ we have $\Sigma(a) = N_{\mathbb{R}}$ and $I_\varnothing(a) = \{i\}$. Moreover, for any $a \in \mathrm{RelInt}(\tilde{\Gamma}_{N_{\mathbb{R}},\{i\}})$ we have that φ_a is a linear function on $N_{\mathbb{R}}$ and without loss of generality we may assume that it vanishes. Then we have $a_j = 0$ for $j \neq i$ and $a_i > 0$, so we see $\Gamma_{N_{\mathbb{R}},\{i\}} = \mathrm{Cone}(\beta_i)$ for β_i in extremal position.
This allows us to reformulate (4.1) to

$$I_\varnothing = \{i; \beta_i \text{ is in extremal position and } \Gamma_{N_{\mathbb{R}},\{i\}} \preceq \Gamma_{\Sigma,I_\varnothing}\}. \tag{4.2}$$

Here we use that for a fan, the intersection of two cones of the fan is a face of both. The condition $\Gamma_{N_{\mathbb{R}},\{i\}} \preceq \Gamma_{\Sigma,I_\varnothing}$ is equivalent to Σ refining $N_{\mathbb{R}}$ (which is always true) and $i \in I_\varnothing$. This immediately shows the inclusion "$\supset$" above.
Conversely assume that $i \in I_\varnothing$. We need to show that β_i is in extremal position. Let $a \in \mathrm{RelInt}(\tilde{\Gamma}_{\Sigma,I_\varnothing})$ and assume that $a \in \mathbb{R}^r_{\geq 0}$. Then $i \in I_\varnothing(a)$ means there exist $\lambda_j \geq 0$ with $\sum_j \lambda_j \nu_j = \nu_i$ and such that $\sum_j \lambda_j(-a_j) > -a_i$. This last inequality immediately implies that $\lambda_i < 1$ and it follows that $\frac{1}{1-\lambda_i}\sum_{j\neq i}\lambda_j\nu_j = \nu_i$. Hence $\nu_i \in \mathrm{Cone}(\nu_1, \dots, \widehat{\nu}_i, \dots, \nu_r)$ and so, again by Lemma 2.4, β_i is in extremal position. □

For $I_\varnothing \subset \{1, \dots, r\}$ let $G_{I_\varnothing}$ be the subgraph of G_ν consisting of chambers of the form $\Gamma_{\Sigma,I'_\varnothing}$ with $I'_\varnothing \subset I_\varnothing$ and the edges connecting such chambers. We also consider the set

$$C_{I_\varnothing} = \bigcup_{\substack{(\Sigma,I'_\varnothing) \text{ admissible} \\ I'_\varnothing \subset I_\varnothing}} \Gamma_{\Sigma,I'_\varnothing} \subset \widehat{G}_{\mathbb{R}}.$$

Then we have the following.

Corollary 4.4. Let $i \in \{1,\dots,r\}$. Then

$$C_{\{1,\dots,r\}\smallsetminus\{i\}} = \mathrm{Cone}(\beta_1,\dots,\widehat{\beta_i},\dots,\beta_r).$$

For $I_\varnothing \subset \{1,\dots,r\}$ the set

$$C_{I_\varnothing} = \bigcap_{i\notin I_\varnothing} C_{\{1,\dots,r\}\smallsetminus\{i\}}$$

is a convex polyhedral cone and the graph $G_{I_\varnothing}$ is connected.

Proof. For the first statement we observe that if β_i is not in extremal position, the statement is a trivial consequence of Proposition 4.3 as there are no chambers with $i \in I_\varnothing$. Hence we may assume that β_i is in extremal position. Now remember that by Theorem 3.20 the secondary fan is the coarsest common refinement of all triangulations of $\boldsymbol{\beta}$. This implies that all cones in Σ_{GKZ} are of the form

$$\Gamma_{\Sigma,I_\varnothing} = \bigcap_{J\in A} \mathrm{Cone}(\boldsymbol{\beta}_J) \subset \widehat{G}_{\mathbb{R}}$$

for A a system of sets of indices such that $(\beta_j)_{j\in J}$ are linearly independent for all $J \in A$. By Proposition 4.3 we have $i \in I_\varnothing$ iff $\beta_i \notin \Gamma_{\Sigma,I_\varnothing}$. But as β_i is in extremal position this is the case if and only if $i \notin J$ for some $J \in A$. But then $\Gamma_{\Sigma,I_\varnothing} \subset \mathrm{Cone}(\boldsymbol{\beta}_J) \subset \mathrm{Cone}(\beta_1,\dots,\widehat{\beta_i},\dots,\beta_r)$. Conversely, one sees easily that $\mathrm{Cone}(\beta_1,\dots,\widehat{\beta_i},\dots,\beta_r)$ is a union of cones of the secondary fan not containing β_i by choosing a triangulation of $\boldsymbol{\beta}\smallsetminus\{\beta_i\}$ and enriching it (via Lemma 3.21) to a triangulation of $\boldsymbol{\beta}$. This shows the other inclusion and the first part of the Corollary.
For the second part of the statement, going to the definitions and writing them out carefully, the statement we have to prove is

$$\bigcup_{I'_\varnothing\subset I_\varnothing} \Gamma_{\Sigma',I'_\varnothing} = \bigcap_{i\notin I_\varnothing}\ \bigcup_{I''_\varnothing\not\ni i} \Gamma_{\Sigma'',I''_\varnothing}.$$

Carefully staring at this expression tells us that the inclusion "$\subset$" is trivial, as for $\Gamma_{\Sigma,I'_\varnothing}$ on the left we may choose $I''_\varnothing = I'_\varnothing$ on the right, using that $i \notin I_\varnothing$ implies $i \notin I'_\varnothing$. For the other inclusion we note that, as the secondary fan is a fan, we have

$$\Gamma_{\Sigma^{12},I^{12}_\varnothing} = \Gamma_{\Sigma^1,I^1_\varnothing} \cap \Gamma_{\Sigma^2,I^2_\varnothing}$$

is again a cone of the secondary fan and a face of $\Gamma_{\Sigma^i,I^i_\varnothing}$, $i = 1,2$, and thus $I^{12}_\varnothing \subset I^1_\varnothing \cap I^2_\varnothing$. This implies that the right side above, after switching the order of intersection and union, is actually a large union of cones $\Gamma_{\Sigma''',I'''_\varnothing}$ obtained as intersection of cones of the form $\Gamma_{\Sigma'',I''_\varnothing}$, $I''_\varnothing \not\ni i$ for $i \notin I_\varnothing$. But as seen above this implies $I'''_\varnothing \subset I_\varnothing$, which finishes the other inclusion.
We see that $G_{I_\varnothing}$ is connected by the same type of argument we applied in Proposition 4.2, using that $C_{I_\varnothing}$ is convex. We note here, that given a chamber $\Gamma_{\Sigma',I'_\varnothing}$ whose interior intersects $C_{I_\varnothing}$, it is already contained in $C_{I_\varnothing}$ as the intersection of all other cones of

the secondary fan with $\mathrm{Int}(\Gamma_{\Sigma',I'_\varnothing})$ is empty. □

While this statement already gives us some more structure for the graph G_ν, we could hope that a stronger statement is true: for given $I_\varnothing$ define the set

$$S_{I_\varnothing} = \bigcup_{\Gamma_{\Sigma,I_\varnothing} \text{ a chamber of } \Sigma_{\mathrm{GKZ}}} \Gamma_{\Sigma,I_\varnothing} \subset \widehat{G}_\mathbb{R}.$$

If these sets were convex cones, we could obtain a very nice and geometric decomposition of the vertex set of G_ν into disjoint subsets $H_{I_\varnothing}$ corresponding to those chambers of the form $\Gamma_{\Sigma,I_\varnothing}$. However, this statement is false. A couter-example is given below in Figure 4.1. This two-dimensional picture should be thought of as representing a slice $x_3 = 1$ in a three-dimensional vector space $\widehat{G}_\mathbb{R}$.

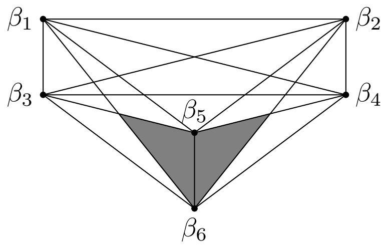


Figure 4.1.: An example for a configuration $\boldsymbol{\beta}$ with $S_{\{6\}}$ not convex

However, under an additional assumption (which will be satisfied in our main series of examples) we can make sure the sets $S_{I_\varnothing}$ are convex polyhedral cones. A vector configuration $\boldsymbol{\beta} \subset \widehat{G}_\mathbb{R}$ will be called **outward convex** if for each i there is at most one cocircuit with $J_+ = \{i\}$. Geometrically this means that when considering the minimal set of half-spaces defining $\mathrm{Cone}(\beta_1, \dots, \widehat{\beta_i}, \dots, \beta_r)$, the vector β_i is contained in all of them except possibly one. Note that this condition has parallels with the "geometric" requirement, which can be formulated to say that there are no co-circuits of $\boldsymbol{\beta}$ with $|J_+| = 1, |J_-| \le 1$.

Theorem 4.5. If $\boldsymbol{\beta}$ is a outward convex vector configuration, the sets $S_{I_\varnothing} \subset \widehat{G}_\mathbb{R}$ are convex polyhedral cones. The subgraphs induced by the sets $H_{I_\varnothing}$ are connected.

Proof. Before giving a rigorous proof, let us give an intuitive explaination how the outward convex condition comes into play. If there are chambers with index set $I_\varnothing$, they are all contained in the convex cone $C_{I_\varnothing}$. This will ensure, that if a chamber $\Gamma_{\Sigma',I'_\varnothing}$ is contained in $S_{I_\varnothing}$ we have $I'_\varnothing \subset I_\varnothing$, i.e. $j \notin I'_\varnothing$ for $j \notin I_\varnothing$. Now for the conditions of the form "$i \in I'_\varnothing$" for $i \in I_\varnothing$ to be satisfied by chambers with a convex union, we use the fact that $\boldsymbol{\beta}$ is an outward convex vector configuration. The chambers satisfying this will exactly be the chambers containing β_i by Proposition 4.3. One can then see, that their union is the closure of $\mathrm{Cone}(\beta_1, \dots, \beta_r) \setminus \mathrm{Cone}(\beta_1, \dots, \widehat{\beta_i}, \dots, \beta_r)$. The fact that

$\boldsymbol{\beta}$ is outward convex ensures that this is either empty or the convex set Cone($\boldsymbol{\beta}_{J_0\cup\{i\}}$ where J_-, J_0, J_+ is the unique cocircuit of $\boldsymbol{\beta}$ with $J_+ = \{i\}$.

In order to prove the claim of the theorem, we will not use the results derived earlier in this section, but give an independent proof, where we will directly use the definition of an outward convex vector configuration.

We first show that $S_{I_\varnothing}$ is convex. As a finite union of closed sets it is closed. Let b, b' be two points in $S_{I_\varnothing}$, then we have to show that for all $t \in [0,1]$ we have $tb+(1-t)b' \in S_{I_\varnothing}$. As $S_{I_\varnothing}$ is closed, it is enough to show that in each neighbourhood of $tb+(1-t)b'$ there is a point of $S_{I_\varnothing}$. But now we can perturb b, b' a little such that they both lie in the interior of chambers with index set $I_\varnothing$ and such that the line segment connecting them only meets (finitely many) walls and otherwise goes through the interior of chambers. Again by closedness of $S_{I_\varnothing}$ it suffices to show $tb+(1-t)b' \in S_{I_\varnothing}$ for all but finitely many t, so in particular we may restrict to those t such that $tb+(1-t)b'$ lies in the interior of a chamber $\Gamma_{\Sigma', I'_\varnothing}$. We then have to show that $I'_\varnothing = I_\varnothing$. But if a, a' are preimages in $\mathbb{R}^r_{\geq 0}$ of b, b' under the map $\gamma_\mathbb{R}$ it suffices to show $I_\varnothing(ta+(1-t)a') = I_\varnothing$. This is the statement we will show now.

By our reduction we already achieved $I_\varnothing(a) = I_\varnothing(a') = I_\varnothing$. Observe that for $i \in \{1, \dots, r\}$ we have

$$i \notin I_\varnothing(\tilde{a}) \iff \text{For all } \lambda_1, \dots, \lambda_r \geq 0 \text{ with } \sum_j \lambda_j \nu_j = \nu_i \text{ we have } \sum_j \lambda_j(-\tilde{a}_j) \leq -\tilde{a}_i.$$

Let $i \notin I_\varnothing = I_\varnothing(a) = I_\varnothing(a')$, then for $\lambda_1, \dots, \lambda_r \geq 0$ with $\sum_j \lambda_j \nu_j = \nu_i$ we have

$$\sum_j \lambda_j (ta_j + (1-t)a'_j) = t\sum_j \lambda_j a_j + (1-t)\sum_j \lambda_j a'_j \geq ta_j + (1-t)a'_j,$$

hence $i \notin I_\varnothing(ta+(1-t)a')$. This holds in general, without the assumption that $\boldsymbol{\beta}$ is a outward convex vector configuration.

Now assume $i \in I_\varnothing$, then we find some $\lambda_j, \lambda'_j \geq 0$ with $\sum_j \lambda_j \nu_j = \nu_i, \sum_j \lambda'_j \nu_j = \nu_i$ satisfying $\sum_j \lambda_j a_j < a_i$ and $\sum_j \lambda'_j a'_j < a'_i$. In general these inequalities will not behave well under convex combination.

But now we note that the set $P_i = \{\lambda \in \mathbb{R}^r_{\geq 0}; \sum_j \lambda_j \nu_j = \nu_i\}$ is a convex polyhedron. The linear functional $\lambda \mapsto \sum_j \lambda_j a_j$ is bounded below by 0 on this polyhedron, so it takes its minimum on P_i at some of its vertices (which must exist as $\mathbb{R}^r_{\geq 0}$ is strongly convex). But those correspond to relations $-\sum_j \lambda_j \nu_j + \nu_i = 0$ with a minimal set of nonzero λ_j, i.e. circuits of $\boldsymbol{\nu}$ with $J_+ = \{i\}$. But as $\boldsymbol{\beta}$ is outward convex, there is at most one such relation. Returning to our situation above, we see that we may modify λ and λ' to both equal this vertex of P_i. Then finally we have

$$\sum_j \lambda_j (ta_j + (1-t)a'_j) = t\sum_j \lambda_j a_j + (1-t)\sum_j \lambda'_j a'_j < ta_j + (1-t)a'_j,$$

We now have that $S_{I_\varnothing}$ is convex. But any finite union of convex polyhedral cones that is itself convex, is already itself a polyhedral cone (namely the convex hull of the finite union of all the generators of the cones). Then, arguing as in Proposition 4.2 we see that $H_{I_\varnothing}$ is a connected graph. □

The theorem above gives us a very nice global structure of the graph of regular triangulations for a geometric vector configuration $\boldsymbol{\nu}$ such that $\boldsymbol{\beta}$ is outward convex. Its vertices decompose into the disjoint sets $H_{I_\varnothing}$, which induce connected subgraphs and can only be nonempty if

$$I_\varnothing \subset \mathcal{I}_\varnothing = \{i; \beta_i \text{ in extremal position}\}$$

by Proposition 4.3.
For $\boldsymbol{\nu}$ geometric we have that every wall between two chambers is either a divisorial or a flipping wall by Theorem 4.1. We see that all edges connecting two vertices both contained in some $H_{I_\varnothing}$ must correspond to flipping walls (as $I_\varnothing$ does not change). Conversely, if there is an edge between a vertex in $H_{I_\varnothing}$ and one in $H_{I'_\varnothing}$ then it corresponds to a divisorial wall and we have $I_\varnothing = I'_\varnothing \dot\cup \{i\}$ (or vice versa).
Now consider what happens, when we contract the subgraphs $H_{I_\varnothing}$ in G_ν. This means we form a new graph $\tilde{G}_\nu$ with the nonempty $H_{I_\varnothing}$ as vertices and an edge between $H_{I_\varnothing}$ and $H_{I'_\varnothing}$ if there are vertices $v \in H_{I_\varnothing}, v' \in H_{I'_\varnothing}$ connected by an edge in G_ν. This graph will canonically be a subgraph of the graph on $\mathcal{P}(\mathcal{I}_\varnothing)$ with edges connecting sets of the form $I_\varnothing$ and $I_\varnothing \dot\cup \{i\}$.
We now give an existence result for edges in G_ν corresponding to divisorial walls.

Proposition 4.6. Let $\boldsymbol{\nu} \subset N_\mathbb{R}$ be a vector configuration and let $\Gamma_{\Sigma, I_\varnothing}$ be a chamber of the secondary fan. If $i \in I_\varnothing$ such that $\nu_i \neq 0$ and such that there is no $j \neq i$ with $\mathrm{Cone}(\nu_i) = \mathrm{Cone}(\nu_j)$, then $\Gamma_{\Sigma, I_\varnothing}$ has a divisorial wall τ, such that on the other side of τ we have the chamber $\Gamma_{\Sigma^*, I_\varnothing \smallsetminus \{i\}}$, where Σ^* is the star-subdivision of Σ at ν_i.

Proof. As $i \in I_\varnothing$ we have that $\beta_i \in \Gamma_{\Sigma, I_\varnothing}$. Choose a point $b_0 \in \mathrm{Int}(\Gamma_{\Sigma, I_\varnothing})$ such that the ray from β_i to b_0 "exits $\Gamma_{\Sigma, I_\varnothing}$ through a facet", i.e. it does not meet a face of $\Gamma_{\Sigma, I_\varnothing}$ of codimension greater than 1. Now consider a preimage $a \in \gamma_\mathbb{R}^{-1}(b_0) \cap \mathbb{R}^r_{\geq 0}$ and note that by adding the vector $\epsilon(1, \ldots, 1)$ for $\epsilon > 0$ small we may assume that $a_j > 0$ for all j. This immediately implies that $\varphi_a(x) > 0$ for $0 \neq x \in C_\nu$. Now look at the path $a(t) = a - te_i$ in $\mathbb{R}^r$ for $t \in [0, a_i]$, whose image is contained in C_β. We see easily that for $t < \varphi_a(\nu_i) + a_i$ we have $\varphi_{a(t)} = \varphi_a$ and hence $\Sigma(a(t)) = \Sigma, I_\varnothing(a(t)) = I_\varnothing$. This means that $\gamma_\mathbb{R}(a(t))$ still lies in the interior of $\Gamma_{\Sigma, I_\varnothing}$. However, for $t = \varphi_a(\nu_i) + a_i$ we have the same fan $\Sigma(a(t)) = \Sigma$ but $I_\varnothing(a(t)) = I_\varnothing \smallsetminus \{i\}$. By assumption we see that $\gamma_\mathbb{R}(a(t))$ now lies in the relative interior of a facet τ of $\Gamma_{\Sigma, I_\varnothing}$. This facet must be of the form $\tau = \Gamma_{\Sigma_0, I_\varnothing \smallsetminus \{i\}}$ and we claim that it is actually a wall, i.e. it is not a facet of C_β. Indeed as $\varphi_a(\nu_i) < 0$ we see that $\varphi_a(\nu_i) + a_i < a_i$ and thus the path $a(t)$ does not leave C_β at $t = \varphi_a(\nu_i) + a_i$. Comparing with Theorem 4.1 and using that by assumption there is no flip with $J_+ = \{i\}$ and $|J_-| \leq 1$, we see that the chamber on the other side is of the desired form. Here we use that a flip on Σ along a circuit satisfying $J_+ = \{i\}, |J_-| \geq 2$ is exactly the star-subdivision at ν_i. □

This result not only shows the existence of edges, but consequently also the existence of other chambers. We obtain the following result, which is Proposition 15.1.6 in [CLS11], as a Corollary to the Proposition 4.6 above.

Corollary 4.7. The secondary fan has a chamber of the form $\Gamma_{\Sigma,\varnothing}$ if and only if $\boldsymbol{\nu}$ is geometric.

Proof. If there exists a chamber of the form $\Gamma_{\Sigma,\varnothing}$ then by Proposition 3.17 there is a bijection

$$\{1,\dots,r\} \setminus \varnothing = \{1,\dots,r\} \to \Sigma(1), i \mapsto \mathrm{Cone}(\nu_i).$$

Thus all the ν_i are nonzero and generate distinct cones, so $\boldsymbol{\nu}$ is geometric.
Conversely, assume $\boldsymbol{\nu}$ is geometric and pick any chamber $\Gamma_{\Sigma,I_\varnothing}$ of the secondary fan. Then we successively apply Proposition 4.6 to the indices $i \in I_\varnothing$. The fact that $\boldsymbol{\nu}$ is geometric ensures that the condition of the Proposition is always satisfied. □

By Corollary 4.4 we know that the union $C_\varnothing$ of all cones in Σ_{GKZ} of the form $\Gamma_{\Sigma,\varnothing}$ is a polyhedral cone and the intersection of all the cones $\mathrm{Cone}(\beta_1,\dots,\widehat{\beta_i},\dots,\beta_r)$. This last statement is also shown in Proposition 15.2.4 of [CLS11]. The cone $C_\varnothing$ is called the **moving cone** of the secondary fan. In Theorem 15.1.10 of [CLS11] it is shown, that for $\boldsymbol{\nu}$ geometric with ν_i a minimal ray generator with respect to the lattice N that if $\Gamma_{\Sigma,\varnothing}$ is a chamber of the secondary fan, the moving cone is isomorphic to the closure of the cone generated by movable Cartier divisors on X_Σ. We don't want to go into further details here.
We will study the graph of regular triangulations for our main series of examples in the next chapter. It will be important for our understanding of the quotient varieties, that can occur there and how they are related.
One result that we want to mention for the general case, is that for $\boldsymbol{\nu}$ geometric, it was shown in [BGS93] that the secondary fan is the normal fan of a polyhedron. As the chambers of the secondary fan then correspond to the vertices of the polyhedron, and two vertices are connected by an edge of the polyhedron iff the chambers share a common wall, we see that G_ν is the graph of bounded edges of a polyhedron. We will also not use this result later.

5. The secondary fan of $\mathbf{Bl}_{n+1}\mathbb{P}^n$

In this Chapter we will investigate our main series of examples, the blow-up $\mathrm{Bl}_{n+1}\mathbb{P}^n$ of $n+1$ points in general position on the variety $\mathbb{P}^n$. This is a toric variety and as a first step, we will represent it as the quotient of a toric action $(\mathbb{C}^*)^{n+2} \curvearrowright \mathbb{C}^{2n+2}$. We will then investigate the other possible quotients arising for this action by applying the techniques and results of the previous chapters.
In Section 5.1 the quotient description of $\mathrm{Bl}_{n+1}\mathbb{P}^n$ is constructed and we make some elementary observations concerning the vector configurations associated with it. In Section 5.2 we cover some low-dimensional examples (for $n = 1, 2, 3$) in detail, to give an intuition about the combinatorics of the secondary fan. Finally in Section 5.3 we treat the general case, find the asymptotic of a family of chambers that we found and give an upper bound on the total number of chambers in the secondary fan.

5.1. First properties

As a first step we have to identify a fan $\Sigma_{\mathrm{Bl}_{n+1}\mathbb{P}^n}$ whose toric variety is $\mathrm{Bl}_{n+1}\mathbb{P}^n$. For this we start with the variety $\mathbb{P}^n$ and successively blow up points in general linear position. The corresponding fan will give us a quotient description for $\mathrm{Bl}_{n+1}\mathbb{P}^n$, corresponding to Gale dual vector configurations $\boldsymbol{\nu}$ and $\boldsymbol{\beta}$. We will then analyse the oriented matroid and the symmetries of these configurations.

5.1.1. The toric variety $\mathbb{P}^n$ revisited

We will show here that $\mathbb{P}^n$ is the toric variety of the fan $\Sigma_{\mathbb{P}^n}$ given in Section 2.3.3. Consider the well-known description of $\mathbb{P}^n$ as the quotient of $\mathbb{C}^{n+1}$ under the action of $G = \mathbb{C}^*$ by

$$t(x_0, \dots, x_n) = (tx_0, \dots, tx_n).$$

This is exactly the type of action considered in Chapter 3. We see that we obtain this particular action by choosing $\beta_1 = \beta_2 = \dots = \beta_n = 1 \in \mathbb{R} = \widehat{G}_{\mathbb{R}}$. One possible (and natural) choice of Gale dual $\boldsymbol{\nu}$ for this $\boldsymbol{\beta}$ is

$$\nu_1 = e_1, \nu_2 = e_2, \dots \nu_n = e_n, \nu_{n+1} = -e_1 - e_2 \dots - e_n,$$

where $e_1, \dots, e_n$ is the standard basis of $N_{\mathbb{R}} = \mathbb{R}^n$ with $N = \mathbb{Z}^n$. The secondary fan of $\boldsymbol{\nu}$ is a fan with support $C_\beta = \mathbb{R}_{\geq 0}$, so it must be the fan $\{\{0\}, C_\beta\}$. The cone $\{0\}$ corresponds to the fan consisting of the entire space $N_{\mathbb{R}}$ and correspondingly the quotient is a single point. On the other hand the chamber C_β corresponds to the simplicial fan $\Sigma_{\mathbb{P}^n}$ which consisted of all cones spanned by proper subsets of $\{\nu_1, \dots, \nu_{n+1}\}$ and the

set $I_\varnothing = \varnothing$.
To visualize this, note that the vector $a = (0, 0, \ldots, 1) \in \mathbb{R}^{n+1}$ maps to the interior of C_β. Hence we obtain the desired fan by placing the vectors $e_1, \ldots, e_n$ on height 0 in $N_\mathbb{R} \times \mathbb{R}$ and the vector $-e_1 - \ldots - e_n$ on height -1, taking the convex cone of this set in $N_\mathbb{R} \times \mathbb{R}$ and projecting its upper facets to $N_\mathbb{R}$. From the description of the irrelevant ideal $B(\Sigma, I_\varnothing)$ given in equation (3.13) one sees that $(\mathbb{C}^r)^{ss} = \mathbb{C}^r \smallsetminus \{0\}$, so indeed we obtain the desired quotient $\mathbb{P}^n = (\mathbb{C}^r \smallsetminus \{0\}) /\!\!/ \mathbb{C}^*$.
We remind the reader that via the orbit-cone correspondence, the maximal cones $\sigma_i = \mathrm{Cone}(e_0, e_1, \ldots, \widehat{e_i}, \ldots, e_n)$ of $\Sigma_{\mathbb{P}^n}$ correspond to the $n+1$ coordinate points in $\mathbb{P}^n$.

5.1.2. A quotient description of $\mathrm{Bl}_{n+1}\mathbb{P}^n$

To obtain the toric variety $\mathrm{Bl}_{n+1}\mathbb{P}^n$, we will now successively blow up the coordinate points of $\mathbb{P}^n$. They are of course in general linear position and they are fixed-points of the torus action. Remember from Section 2.3.3 that the blow-up of torus fixed points in X_Σ corresponds to the star-subdivision of the corresponding maximal cones σ in the fan Σ. Now we apply this procedure to the $n+1$ coordinate points in $\mathbb{P}^n$. We see that this corresponds to star-subdivisions in all maximal cones σ_i. This implies that the resulting simplicial fan $\Sigma_{\mathrm{Bl}_{n+1}\mathbb{P}^n}$ does not depend on the order in which points are blown up. For $\sigma_0 = \mathrm{Cone}(e_1, \ldots, e_n)$ the newly introduced ray generator is $e_1 + \ldots + e_n$. For $i = 1, \ldots, n$ the ray generator is

$$(-e_1 - e_2 - \ldots - e_n) + e_1 + \ldots \widehat{+e_i} \ldots + e_n = -e_i.$$

So the vectors $\nu_1, \ldots, \nu_{2n+2} \in \mathbb{R}^n$ with

$$\begin{aligned} \nu_1 &= -\nu_{n+2} = e_1, \\ \nu_2 &= -\nu_{n+3} = e_2, \\ &\vdots \\ \nu_n &= -\nu_{2n+1} = e_n, \\ \nu_{n+1} &= -\nu_{2n+2} = -(e_1 + e_2 + \ldots + e_n) \end{aligned}$$

are the minimal generators of the rays in $\Sigma_{\mathrm{Bl}_{n+1}\mathbb{P}^n}(1)$. Let $\boldsymbol{\beta}$ be a Gale dual vector configuration to $\boldsymbol{\nu}$. Our next result shows that one of the chambers of the corresponding secondary fan will have $\Sigma_{\mathrm{Bl}_{n+1}\mathbb{P}^n}$ as its corresponding fan.

Proposition 5.1. Let $\boldsymbol{\nu} \subset N_\mathbb{R}$ be a vector configuration with $C_\nu = \mathrm{Cone}(\boldsymbol{\nu}) = N_\mathbb{R}$ with corresponding secondary fan Σ_{GKZ} and let Σ be a generalized fan in $N_\mathbb{R}$. Then there exists a chamber $\Gamma_{\Sigma, I_\varnothing} \in \Sigma_{\mathrm{GKZ}}$ (for some $I_\varnothing \subset \{1, \ldots, r\}$) corresponding to Σ if and only if Σ is a complete, simplicial fan with $\Sigma(1) \subset \{\mathrm{Cone}(\nu_i); i = 1, \ldots, r\}$ and X_Σ is projective.

Proof. By Proposition 14.4.1 from [CLS11] we have that $(\Sigma, I_\varnothing)$ is an admissible pair if and only if

- $|\Sigma| = C_\nu = N_\mathbb{R}$, i.e. Σ is complete. Note that by Theorem 3.1.19 from [CLS11] this is equivalent to X_Σ being compact in the classical topology.
- X_Σ is semiprojective. This is a property of a variety introduced in [CLS11]. In Proposition 7.2.9. of [CLS11] a characterisation of semiprojectivity is proven, which shows that in our case (as $|\Sigma| = N_\mathbb{R}$ is full-dimensional convex) this is equivalent to X_Σ being quasi-projective.
- $\sigma = \mathrm{Cone}(\nu_i; \nu_i \in \sigma, i \notin I_\varnothing)$ for all $\sigma \in \Sigma$.

If in addition we require $(\Sigma, I_\varnothing)$ to correspond to a chamber of the secondary fan then by Proposition 3.17 this is equivalent to adding the condition

- Σ is simplicial and there is a bijection $\{1, \dots, r\} \smallsetminus I_\varnothing \to \Sigma(1), i \mapsto \mathrm{Cone}(\nu_i)$.

Now Σ simplicial implies that all cones in Σ are spanned by their rays, so by enlarging $I_\varnothing$ if necessary we may always reach that $i \mapsto \mathrm{Cone}(\nu_i)$ defines the desired bijection. Hence the last condition reduces to Σ being simplicial.
Finally note that being compact in the classical topology and quasi-projective is equivalent to being projective (this follows easily from Lemma 7.1 in [Šaf94b]). As Σ is a fan, the third property must only be checked for rays $\sigma \in \Sigma(1)$ and thus $\Sigma(1) \subset \{\mathrm{Cone}(\nu_i); i = 1, \dots, r\}$ is a necessary and sufficient condition. □

Applying above Proposition to our situation we see that there will be a chamber of the secondary fan corresponding to the variety $\mathrm{Bl}_{n+1}\mathbb{P}^n$ as desired. But we even get more: any projective toric variety whose simplicial fan we can build using only the rays generated by the ν_i above will appear as the quotient of one of the chambers. Together with the symmetries of the vector configuration $\boldsymbol{\nu}$ below this will give us a decent number of chambers just by guessing (families of) such varieties.
To obtain an explicit description of the torus action corresponding to the vectors $\boldsymbol{\nu}$, we have to compute the set of Gale dual vectors $\boldsymbol{\beta}$. These generate the relations among the ν_i. Observe that we naturally have the relations $\nu_i = -\nu_{i+n+1}$ for $i = 1, \dots, n+1$. A last linearly independent relation is given by $\nu_{n+2} + \nu_{n+3} + \dots + \nu_{2n+2} = 0$. From now on we will identify the vector space $M_\mathbb{R}$ with $\mathbb{R}^n$ and $\widehat{G}_\mathbb{R}$ with $\mathbb{R}^{n+2}$. Then the corresponding exact sequence

$$0 \to M_\mathbb{R} \xrightarrow{B} \mathbb{R}^{2n+2} \xrightarrow{A} \widehat{G}_\mathbb{R} \to 0,$$

is given by the matrices

$$B = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ -1 & \dots & -1 \\ -1 & & 0 \\ & \ddots & \\ 0 & & -1 \\ 1 & \dots & 1 \end{pmatrix}, A = \begin{pmatrix} 1 & & 0 & 1 & & 0 \\ & \ddots & & & \ddots & \\ 0 & & 1 & 0 & & 1 \\ 0 & \dots & 0 & 1 & \dots & 1 \end{pmatrix}.$$

From the entries of A we can then read off the desired action $(\mathbb{C}^*)^{n+2} \curvearrowright \mathbb{C}^{2n+2}$ as given by

$$\begin{aligned}&(s_1,\dots,s_{n+2}).(z_1,\dots z_{2n+2})\\=&(s_1z_1,s_2z_2,\dots,s_{n+1}z_{n+1},s_1s_{n+2}z_{n+2},s_2s_{n+2}z_{n+3},\dots,s_{n+1}s_{n+2}z_{2n+2}).\end{aligned}$$

While the choice and arrangement of A above will be very convenient when computing the secondary fan, a different arrangement can produce a somewhat more elegant description of an isomorphic action. For this we name the current coordinates on $\mathbb{C}^{2n+2}$ by $x_1, x_2, \dots, x_{n+1}, y_1, y_2, \dots, y_{n+1}$. By a permutation of coordinates, we reorder them to $x_1, y_1, x_2, y_2, \dots, x_{n+1}, y_{n+1}$. We also write coordinates in $(\mathbb{C}^*)^{n+2}$ by $t_1, \dots, t_{n+1}, \lambda$. Then the action has the form

$$(t_1,\dots,t_{n+1},\lambda).(x_1,y_1,\dots,x_{n+1},y_{n+1}) = (t_1x_1,\lambda t_1y_1,\dots,t_{n+1}x_{n+1},\lambda t_{n+1}y_{n+1}).$$

5.1.3. The vector configurations $\boldsymbol{\nu}$ and $\boldsymbol{\beta}$

In this section we will try to give a first insight into the geometry of the vector configurations $\boldsymbol{\nu}$ and $\boldsymbol{\beta}$. We analyse their symmetries and compute their associated circuits and cocircuits.

Visualizing the vectors in $\boldsymbol{\nu}$ is very straightforward: they are the positive and negative standard basis vectors of $\mathbb{R}^n$ together with $\pm(1,1,\dots,1)^t$. To visualize $\boldsymbol{\beta} \subset \mathbb{R}^{n+2}$ we write $\mathbb{R}^{n+2} = \mathbb{R}^{n+1} \times \mathbb{R}$. Then $\beta_1, \dots, \beta_{n+1}$ are the vertices of the standard n-simplex in $\mathbb{R}^{n+1} = \mathbb{R}^{n+1} \times \{0\}$ and $\beta_{n+2}, \dots, \beta_{2n+2}$ are those vertices translated by 1 along the last coordinate. We also note that the cone C_β spanned by $\boldsymbol{\beta}$ is strongly convex as it is a subset of the strongly convex cone $\mathrm{Cone}(e_1, \dots, e_{n+2})$. This gives us a way to visualize $\boldsymbol{\beta}$ "in one dimension lower". For this note that $\boldsymbol{\beta}$ is contained in the affine hyperplane given by $x_1 + x_2 + \dots + x_{n+1} = 1$. When we consider $\boldsymbol{\beta}$ as part of this $(n+1)$-dimensional space, it becomes the prism over the standard n-simplex. In this picture cones spanned by subsets of $\boldsymbol{\beta}$ become polytopes spanned by the corresponding points. This intuition will be very useful, especially for the low-dimensional examples. Note also that the vector configuration $\boldsymbol{\nu}$ is geometric for $n \geq 2$.

Now we analyse the symmetries of the configurations. The correct group actions to consider are the actions of $\mathrm{GL}(n, \mathbb{Z})$ on $\mathbb{R}^n$ and $\mathrm{GL}(n+2, \mathbb{Z})$ on $\mathbb{R}^{n+2}$. This is because $\Sigma_{\mathrm{Bl}_{n+1}\mathbb{P}^n}$ is only unique up to the action of the first group. Similarly when $\boldsymbol{\nu}$ is chosen, $\boldsymbol{\beta}$ is only unique up to the action of the second group.

It is clear that by permutations of the coordinates of $\mathbb{R}^n$ we can permute the vectors $\nu_1, \dots, \nu_n$ freely, inducing a corresponding permutation of the vectors $\nu_{n+2}, \dots, \nu_{2n+1}$ and leaving $\nu_{n+1} = -\nu_{2n+2}$ fixed. Another easy operation is the multiplication by -1, which induces the permutation τ sending ν_i to ν_{i+n+1} and vice versa for $i = 1, \dots, n+1$. However these are not all symmetries. Consider the linear map $L_i : \mathbb{R}^n \to \mathbb{R}^n$ determined by

$$L_i e_j = \begin{cases} e_j & \text{for } i \neq j \\ \nu_{n+1} = -e_1 - \dots - e_n & \text{for } i = j \end{cases}$$

Then one sees that $L_i\nu_{n+1} = e_i = -L_i\nu_{2n+2}$. Hence we can transpose ν_i and ν_{n+1} for $i = 1, \dots n$ and thus we can freely permute the first $n+1$ vectors in $\boldsymbol{\nu}$, while the last $n+1$ vectors undergo the same permutation. We claim that together with the permutation τ this generates the symmetry group for $n \geq 2$.

Proposition 5.2. For $n \geq 2$ the symmetry group of $\boldsymbol{\nu}$ is $S_{n+1} \times \{1, -1\}$, where the action is given by

$$\begin{aligned}
(\sigma, 1)\nu_i &= \nu_{\sigma(i)} && \text{for } 1 \leq i \leq n+1 \\
(\sigma, 1)\nu_i &= \nu_{\sigma(i-(n+1))+n+1} && \text{for } n+2 \leq i \leq 2n+2 \\
(\mathrm{id}, -1)\nu_i &= \nu_{i+n+1} && \text{for } 1 \leq i \leq n+1 \\
(\mathrm{id}, -1)\nu_i &= \nu_{i-(n+1)} && \text{for } n+2 \leq i \leq 2n+2.
\end{aligned}$$

Proof. We have seen above that S_{n+1} and $\{1, -1\}$ are subgroups of the symmetry group of $\boldsymbol{\nu}$ and it is easy to verify that their intersection is trivial and that they commute. Hence it suffices to show that all symmetries are of above type.
Assume we have a linear isomorphism $A \in \mathrm{GL}(n, \mathbb{Z})$ sending $\boldsymbol{\nu}$ to itself. It is of course determined by its values on $\nu_1, \dots, \nu_n$. Consider the set $S = \{A\nu_1, \dots, A\nu_{n+1}\}$. As A is an isomorphism and $n \geq 2$, no two vectors in S are linearly dependent. This implies that for $i = 1, \dots, n+1$ we have either $\nu_i \in S$ or $-\nu_i \in S$. Now additionally we know that the sum of the elements in S is zero. But one quickly checks that this is only possible for $S = \{\nu_1, \dots, \nu_{n+1}\}$ or $S = \{-\nu_1, \dots, -\nu_{n+1}\}$. Thus we see that A induces a permutation of $\nu_1, \dots, \nu_{n+1}$, possibly followed by swapping ν_i and ν_{i+n+1}. □

We note that a similar argument to above shows that the symmetry group of $\boldsymbol{\beta}$ is also $S_{n+1} \times \{1, -1\}$, acting ("on the indices") the same way as above. We want to remark in addition that this is also the symmetry group of the rays generated by the elements of $\boldsymbol{\nu}$ and $\boldsymbol{\beta}$, respectively. One sees this as, by assumption, the lattices M and N are left invariant under the action of $\mathrm{GL}(n, \mathbb{Z})$ and the generators of rays rational with respect to these lattices are unique.
As a last result before looking at some concrete examples (for n small), we compute the oriented matroid of the vector configurations $\boldsymbol{\nu}$ and $\boldsymbol{\beta}$. Especially the cocircuits of $\boldsymbol{\beta}$ will be important later, as they describe the possible wall-crossings of the secondary fan.

Lemma 5.3. For $i, j \in \{1, \dots, n+1\}$ with $i \neq j$ let $\mathrm{C}(i,j) \in \{-, 0, +\}^{2n+2}$ with

$$\mathrm{C}(i,j)_k = \begin{cases} + & \text{for } k = i, j+n+1 \\ - & \text{for } k = j, i+n+1 \\ 0 & \text{otherwise} \end{cases}.$$

Then the circuits of the vector configuration $\boldsymbol{\beta}$ are given by

$$\mathcal{C}(\boldsymbol{\beta}) = \{\mathrm{C}(i,j); i, j \in \{1, \dots, n+1\}, i \neq j\}.$$

There are two types of cocircuits: for $S \subset \{1,\dots,n+1\}$ let $S^c = \{1,\dots,n+1\} \setminus S$. Then we define $\mathrm{U}(S) \in \{-,0,+\}^{2n+2}$ such that

$$\mathrm{U}(S)_j = \begin{cases} 0 & \text{for } j \in S \text{ or } j \in S^c + n + 1 \\ + & \text{for } j \in S^c \\ - & \text{for } j \in S + n + 1 \end{cases}.$$

On the other hand, for $i \in \{1,\dots,n+1\}$ we have

$$\mathrm{U}(i)_j = \begin{cases} + & \text{for } j = i \text{ or } j = i + n + 1 \\ 0 & \text{otherwise} \end{cases}.$$

Then the cocircuits of $\boldsymbol{\beta}$ are given by

$$\mathcal{C}^*(\boldsymbol{\beta}) = \{\pm\mathrm{U}(S); S \subset \{1,\dots,n+1\}\} \cup \{\pm\mathrm{U}(i); i \in \{1,\dots,n+1\}\}.$$

As a consequence, we see that $\boldsymbol{\beta}$ is outward convex.

Proof. Assume we have $a \in \mathbb{R}^{2n+2}$ giving some linear combination $\sum_{k=1}^{2n+2} a_k\beta_k$ of the vectors in $\boldsymbol{\beta}$. Then $\sum_{k=1}^{2n+2} a_k\beta_k = 0$ is equivalent to $a_k = -a_{k+n+1}$ for $k = 1,\dots,n+1$ and $\sum_{k=n+2}^{2n+2} a_k = 0$. Any $a \neq 0$ satisfying this must have at least a positive and a negative entry among the last $n+1$ components and correspondingly a negative and positive entry at positions j,i among the first $n+1$ components. On the other hand we can put $a_i = 1, a_j = -1, a_{i+n+1} = -1, a_{j+n+1} = 1$ and $a_k = 0$ otherwise, which satisfies $\sum_{k=1}^{2n+2} a_k\beta_k = 0$. This gives exactly the circuit $\mathrm{C}(i,j)$.
Now assume we have an oriented hyperplane $H \subset \mathbb{R}^{n+2}$ such that $H \cap \boldsymbol{\beta}$ spans H. Let $S = \{i \in \{1,\dots,n+1\}; \beta_i \in H\}$ and $S' = \{i \in \{1,\dots,n+1\}; \beta_{i+n+1} \in H\}$. For H being spanned by the vectors in $\boldsymbol{\beta}$ contained in it it is necessary that $|S|+|S'| \geq n+1$. Assume first that S and S' are disjoint. Then necessarily $S' = S^c$ and a quick calculation shows that H is given by the equation $(\sum_{k \in S^c} x_k) - x_{n+2} = 0$. Then depending on the orientation of H, the vectors $\beta_j, j \in S^c$ have a positive sign and the vectors $\beta_{j+n+1}, j \in S$ have a negative sign. This corresponds to the sign vector $\mathrm{U}(S)$.
On the other hand, if $k \in S \cap S$ then $e_{n+2} = \beta_{k+n+1} - \beta_k \in H$ and thus $\beta_j \in H$ implies $\beta_{j+n+1} = \beta_j + e_{n+2} \in H$ and vice versa. This implies $S \subset S'$ and $S' \subset S$, so $S = S'$. Then the maximal such S are of the form $\{1,\dots,n+1\} \setminus \{i\}$ and they are realized by the hyperplane $x_i = 0$. This corresponds to the sign vector $\pm\mathrm{U}(i)$. □

5.2. Examples in low dimensions

In the following Sections we will discuss thoroughly some low-dimensional cases for the general quotient construction of $\mathrm{Bl}_{n+1}\mathbb{P}^n$ above. This serves as building up intuition for some results holding in general dimension. We want to exclude the case $n = 0$ here. Here trying to blow up the unique point in $\mathbb{P}^0$ does not change anything on the level of varieties.

5.2.1. $n = 1$

The case $n = 1$ still shows some pathologies vanishing for bigger n. We note that at least the two coordinate points $[1 : 0]$ and $[0 : 1]$ are in general linear position, but blowing them up does not change the variety.
Still it is possible to define the vectors $\boldsymbol{\nu}$ and $\boldsymbol{\beta}$ as described above. We obtain the corresponding matrices

$$B = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Thus we have $\nu_1 = \nu_4 = 1 \in \mathbb{R}$, $\nu_2 = \nu_3 = -1 \in \mathbb{R}$. This behaviour is special as for $n \geq 2$ the vectors in $\boldsymbol{\nu}$ will all be distinct (and even generate distinct rays in $N_{\mathbb{R}}$). From this it follows that in the case $n = 1$ all admissible pairs $(\Sigma, I_\varnothing)$ corresponding to chambers of the secondary fan will satisfy

$$\Sigma = \{\{0\}, \mathbb{R}_{\geq 0}, \mathbb{R}_{\leq 0}\}, (1 \notin I_\varnothing \text{ or } 4 \notin I_\varnothing) \text{ and } (2 \notin I_\varnothing \text{ or } 3 \notin I_\varnothing).$$

Looking at $\boldsymbol{\beta}$ we already have an interesting geometry. When we regard it in the affine plane $x_1 + x_2 = 1$ it is the prism over a 1-simplex, i.e. a rectangle. Drawing all lower-dimensional cones spanned by vectors in $\boldsymbol{\beta}$ we see that the secondary fan has exactly four chambers.

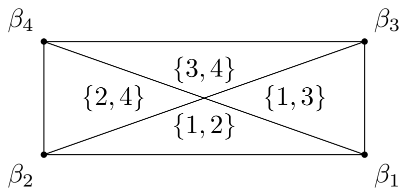


Figure 5.1.: The chambers of the secondary fan for $\mathrm{Bl}_2\mathbb{P}^1$ with $I_\varnothing$ indicated

We see that crossing the hyperplane through β_2 and β_3 from the right to the left amounts to "blowing down" $\boldsymbol{\nu}_4$ and "blowing up" $\boldsymbol{\nu}_1$, i.e. adding 4 to $I_\varnothing$ and removing 1.

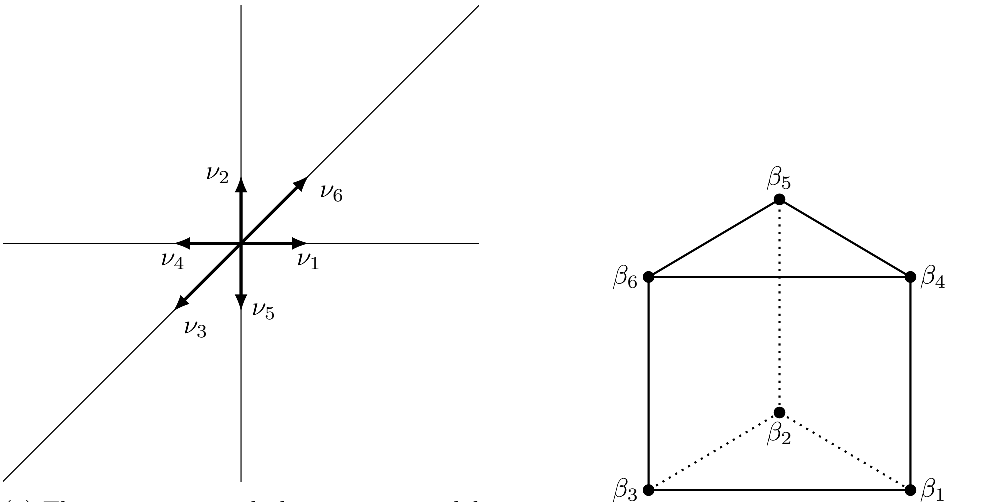
5.2.2. $n = 2$

The case $n = 2$ will in some sense be optimal for our understanding, as we will be able to compute and picture the secondary fan as well as the fans describing the different quotient varieties by hand. The structure of the secondary fan will already be quite rich but we will be able to give a full description of all quotient varieties corresponding to its chambers. There is however one type of wall-crossing, namely a genuine flip,

which does not occur until $n = 3$.
As a first step we write down the matrices containing the information about $\boldsymbol{\nu}$ and $\boldsymbol{\beta}$. They are

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \\ -1 & 0 \\ 0 & -1 \\ 1 & 1 \end{pmatrix}, A = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

The vectors $\nu_i \in \mathbb{R}^2$ are pictured in Figure 5.2 below on the left. The vectors β_i form a prism over a regular triangle contained in a hyperplane of $\mathbb{R}^4$ and are pictured on the right.

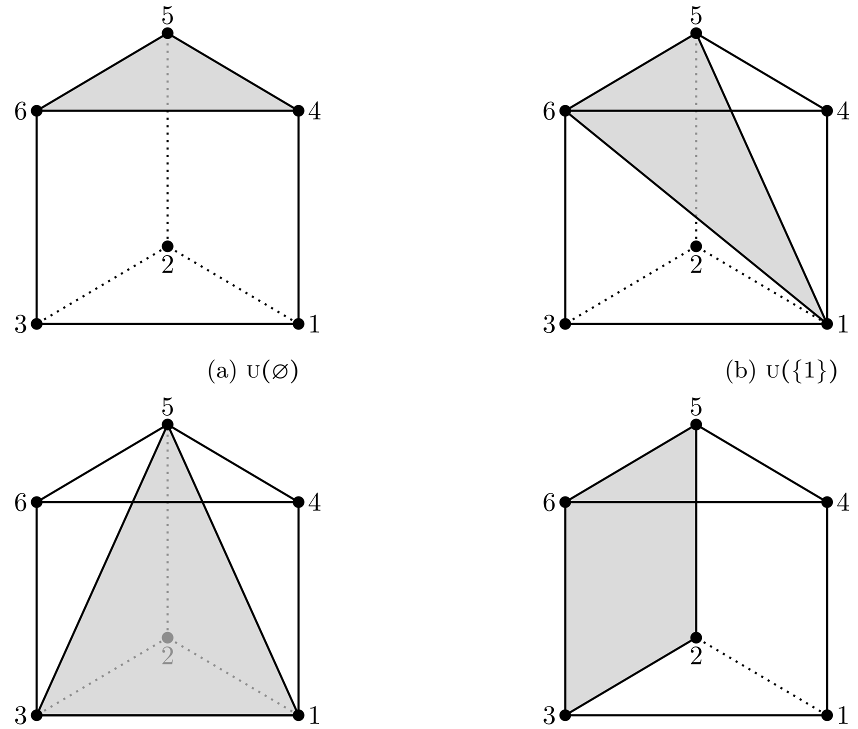


(a) The vectors ν_i and the rays spanned by them

(b) The vectors β_i and their convex hull

Figure 5.2.: The vector configurations $\boldsymbol{\nu}$ and $\boldsymbol{\beta}$

Now to obtain the secondary fan we have to draw all cones of dimension ≤ 3 spanned by subsets of $\boldsymbol{\beta}$. We will also link them to the corresponding cocircuits of $\boldsymbol{\beta}$ defined in Lemma 5.3. The cones that will form walls in the secondary fan are exactly those spanned by β_i, β_j on the lower level of the triangle and β_{k+3} on the upper level for $\{i,j,k\} = \{1,2,3\}$ (and vice versa). They correspond to the cocircuit $\mathrm{U}(\{i,j\})$ (or $\mathrm{U}(\{k\})$, respectively). The outer walls of the prism are the upper and lower triangle, corresponding to $\mathrm{U}(\varnothing)$ and $\mathrm{U}(\{1,2,3\})$, and the three outer walls, corresponding to the cocircuits $\mathrm{U}(1), \mathrm{U}(2), \mathrm{U}(3)$. All three types of cones are depicted in Figure 5.3 below; to be more precise, their intersection with the 3-dimensional affine hyperplane $x_1 + x_2 + x_3 = 1$ is shown. For simplicity we only write the numbers of the points, ommitting the β.



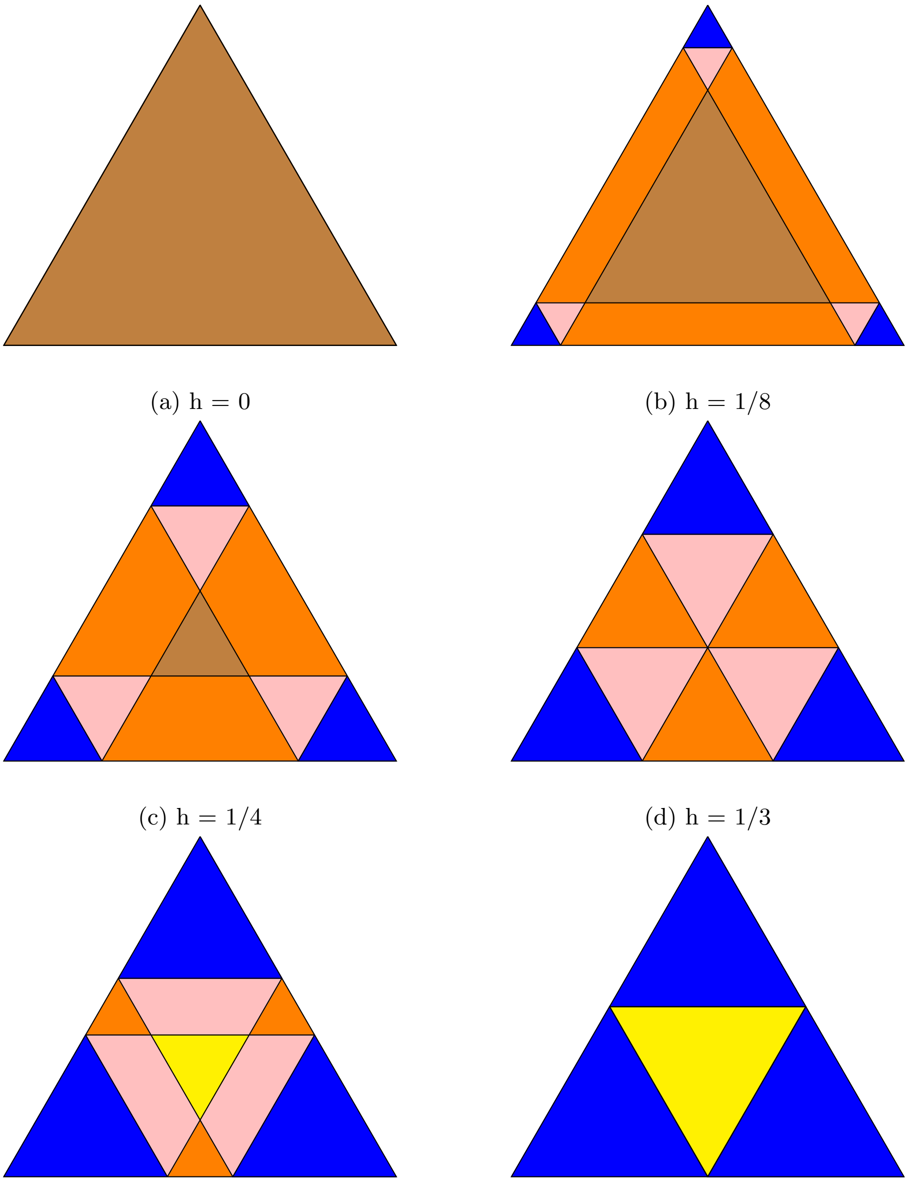
(a) $\mathrm{U}(\varnothing)$ (b) $\mathrm{U}(\{1\})$

(c) $\mathrm{U}(\{1,3\})$ (d) $\mathrm{U}(1)$

Figure 5.3.: Examples of codimension 1 cones spanned by subsets of $\boldsymbol{\beta}$ and the corresponding cocircuits

Now trying to draw a three-dimensional picture involving all the chambers of the fan is quite difficult. Instead we will draw two-dimensional slices through the triangular prism at different heights h. The colours of the chambers below are chosen to indicate already which quotient variety they will produce (see Figure 5.5). For symmetry reasons the picture for a given h is the same as for $1-h$. Therefore we have only shown pictures for $h \leq \frac{1}{2}$.

Now we come to the different fans in $N_{\mathbb{R}}$ associated to the chambers above. For reasons of symmetry we can see that all chambers depicted in the same colour will have isomorphic fans. Note that the fans in $\mathbb{R}^2$ are uniquely determined by their rays. Therefore it suffices to know $I_\varnothing$ for each of the chambers. But $I_\varnothing$ can easily be determined using Corollary 4.4. In Figure 5.5 we give a list of the chambers.
We designate the five different classes of chambers by letters A, B, C, D and E. We give the number of chambers in each equivalence class and the types and numbers



(a) h = 0 (b) h = 1/8

(c) h = 1/4 (d) h = 1/3

(e) h = 5/12 (f) h = 1/2

Figure 5.4.: Slices through the secondary fan Σ_{GKZ} at different heights h

of chambers adjacent to one of those. So the first entry "B(3)" means that each of the two chambers of type A has three neighbouring chambers of type B. Moreover, we identify the quotient variety and give an example of a fan Σ corresponding to the chambers.

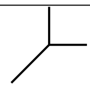
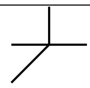
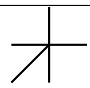
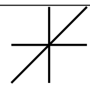
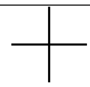
Name X	Colour	# Chambers	Neighbours	Variety V_X	Fan Σ_X
A		2	B(3)	$\mathbb{P}^2$	
B		6	A(1), C(2)	$\mathrm{Bl}_1\mathbb{P}^2$	
C		6	B(2), D(1), E(1)	$\mathrm{Bl}_2\mathbb{P}^2$	
D		1	C(6)	$\mathrm{Bl}_3\mathbb{P}^2$	
E		3	C(2)	$\mathbb{P}^1 \times \mathbb{P}^1$	
		18			

Figure 5.5.: The chamber types of the secondary fan for $\mathrm{Bl}_3\mathbb{P}^2$

Here we see that apart from the "expected" quotient varieties $\mathrm{Bl}_i\mathbb{P}^2$, $i = 0, 1, 2, 3$ we also obtain the variety $\mathbb{P}^1 \times \mathbb{P}^1$. We will see below that in the secondary fan of $\mathrm{Bl}_{n+1}\mathbb{P}^n$ for $n \geq 2$ we always have chambers with quotient $(\mathbb{P}^1)^n$.
From the depiction in Figure 5.4 it is easy to write down the flip graph of regular triangulations for our point configuration $\boldsymbol{\nu}$. It is shown in Figure 5.6.

5.2.3. $n = 3$

For $n = 3$ it is already very hard to compute the different chambers of the secondary fan and their corresponding quotients by hand. This is also due to the fact that the vectors ν_i now live in a three-dimensional space, while the β_i (even after the standard dimension-reduction) correspond to points in four dimensions. However, for the fans corresponding to the different chambers of Σ_{GKZ} we can still give pictures. Here we will see the first occurence of genuine flips (and why they are called "flips").
Again we write down the matrices containing the information about $\boldsymbol{\nu}$ and $\boldsymbol{\beta}$. They

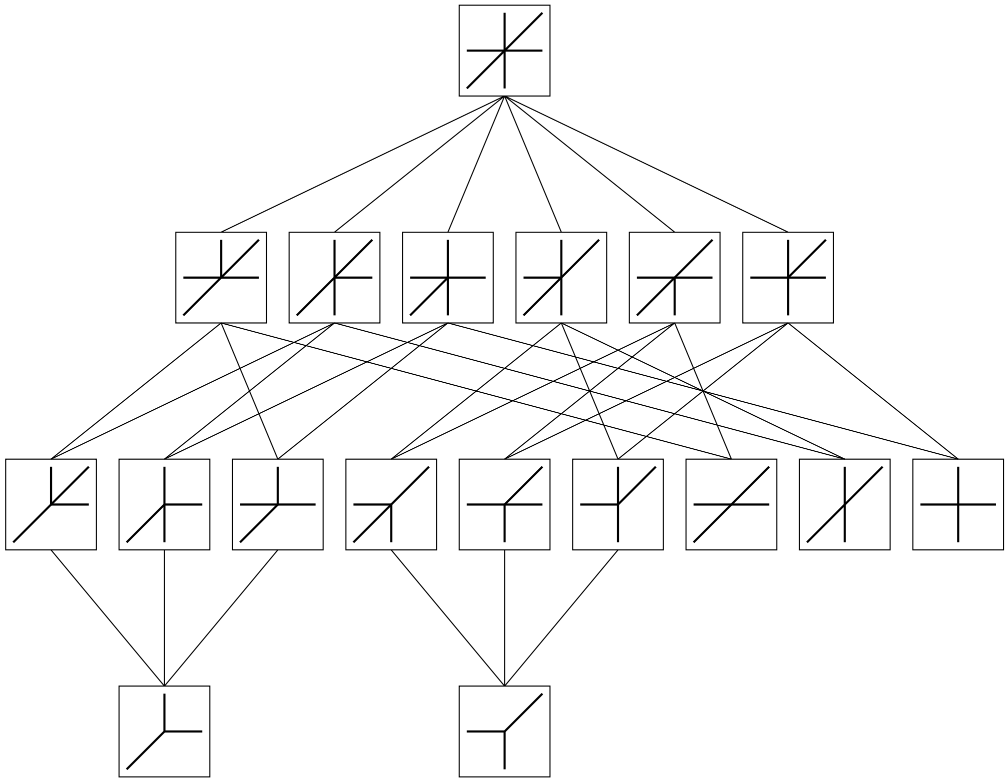
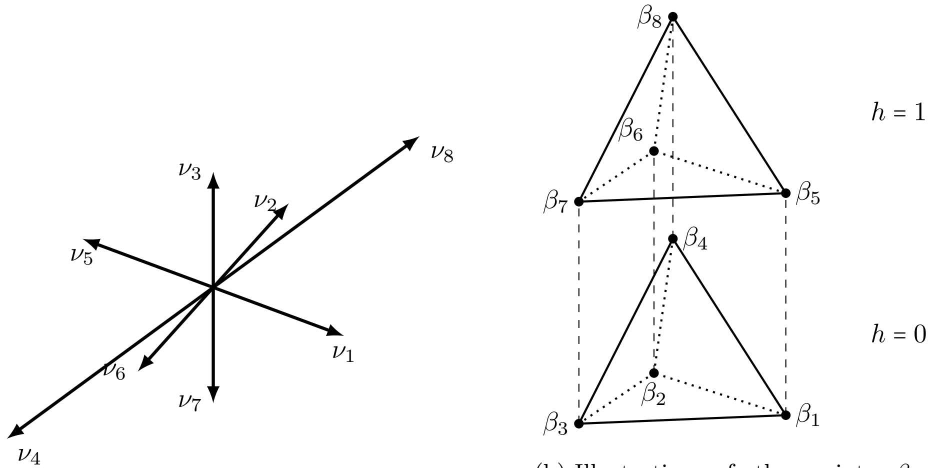


Figure 5.6.: The graph of regular triangulations for $n = 2$

are

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}, A = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

The vectors $\nu_i \in \mathbb{R}^3$ are pictured in Figure 5.7 below on the left. The vectors β_i form a prism over a regular tetrahedron contained in a hyperplane of $\mathbb{R}^5$ and we have tried to picture them on the right.



(a) The vectors ν_i

(b) Illustration of the points β_i in four-dimensional space

Figure 5.7.: The vector configurations $\boldsymbol{\nu}$ and $\boldsymbol{\beta}$

Finding all chambers of the secondary fan by hand would be a rather lengthy task. Instead we use the `Polyhedra` package of `Macaulay2` to find the secondary fan as the coarsest common refinement of all triangulations of $\boldsymbol{\beta}$. It turns out that there are 148 chambers and 14 different equivalence classes of fans corresponding to them. In Appendix A you find pictures for those 14 fans as well as a detailed description of how they were created and how to visualize the corresponding 3-dimensional fans. We want to designate the 14 types of chambers by

$$A_0, A_1, A_2, A_3, A_4, B_1, B_2, B_3, C_1, C_2, D_1, E_1, E_2, E_3.$$

The reason for this naming convention will become apparent in a moment.
Now we want to identify the corresponding toric varieties. We have the following result.

Lemma 5.4. The quotient varieties corresponding to the chambers of the secondary fan of $\mathrm{Bl}_4\mathbb{P}^3$ are of one of the following types:

- For chambers of type A_0, the quotient variety is $\mathbb{P}^n$.
- For chambers of type X_1, where $X \in \{A, B, C, D, E\}$, the quotient variety is a $\mathbb{P}^1$-bundle over the toric variety V_X corresponding to the chamber type X of the secondary fan of $\mathrm{Bl}_3\mathbb{P}^2$(see Figure 5.5).
  These were $V_A = \mathbb{P}^2, V_B = \mathrm{Bl}_1\mathbb{P}^2, V_C = \mathrm{Bl}_2\mathbb{P}^2, V_D = \mathrm{Bl}_3\mathbb{P}^2, V_E = \mathbb{P}^1 \times \mathbb{P}^1$.
- For chambers of type X_k, where $X \in \{A, B, C, D, E\}$, $k \geq 2$, the quotient variety is the blow-up of $k-1$ points on the variety corresponding to the chamber X_1.

Proof. Our proof will be based on the pictures of the corresponding fans in Appendix A.
For the chamber type A_0 we see that we have the fan with rays $-e_1, -e_2, -e_3, e_1 + e_2 + e_3$ and the chambers are the cones spanned by three of the vectors. Under the isomorphism $\mathbb{R}^3 \to \mathbb{R}^3, x \mapsto -x$ this fan is mapped to the standard fan $\Sigma_{\mathbb{P}^3}$ of $\mathbb{P}^3$.
To identify the quotient variety for chambers of the form X_1, the reader should be familiar with the paragraph about products and fibre bundles in Section 2.3.3. We will show for chamber C_1 that its fan Σ is split by a fan isomorphic to Σ_C and $\Sigma_0 = \Sigma_{\mathbb{P}^1}$ for the map

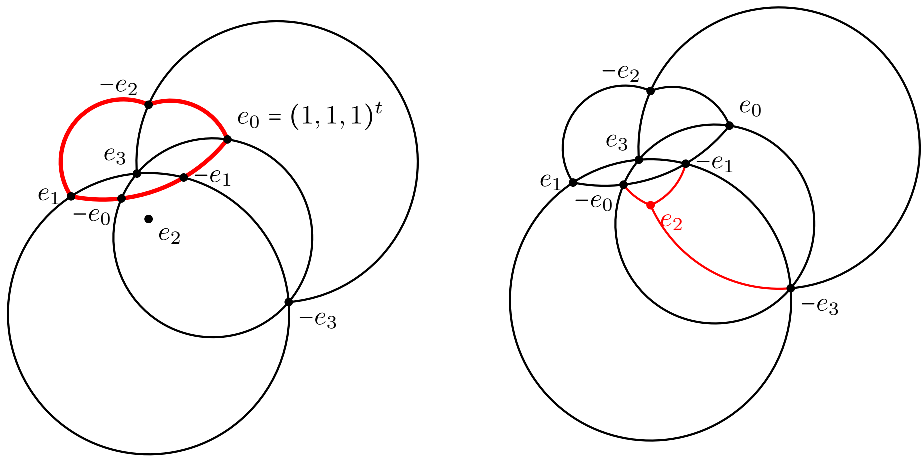
$$\phi : \mathbb{Z}^3 \to \mathbb{Z}^2, (x, y, z) \mapsto (x, y).$$

The proof for the other chambers is similar.
In Figure 5.8a we have pictured a fan Σ from a chamber of type C_1. We identify

$$\Sigma_0 = \{\mathrm{Cone}(-e_3), \{0\}, \mathrm{Cone}(e_3)\} = \Sigma \cap \mathbb{R}e_3.$$

We have also indicated the subfan $\widehat{\Sigma} \subset \Sigma$ from the definition of splitting of fans. Then one can easily check that Σ is split by a fan isomorphic to Σ_C and Σ_0.
For the last part of the statement we simply observe that the fans of the chambers X_k for $k \geq 2$ are obtained from those for X_1 by performing $k-1$ star-subdivisions of smooth cones. As an example see Figure 5.8b, where a star-subdivision at e_2 has been performed on the fan in Figure 5.8a. ☐

While it is clear how to obtain the varieties of type X_k from X_1 for $k \geq 2$, we have not yet characterized the varieties of type X_1 in a satisfactory manner. By the last Lemma, we know that they are toric fibre bundles over the quotient varieties arising from the case $n = 2$. But luckily they are all projective bundles. This was the last type of construction mentioned in Section 2.3.3, where we had a vector bundle of rank $n+1$ over a toric variety and from this obtained a $\mathbb{P}^n$-bundle by using the glueing data of the vector bundle to glue together sets of the form $U_i \times \mathbb{P}^n$.
To identify the varieties of type X_1 as projective bundles, we compare their fans with the fan constructed for such toric varieties in Proposition 7.3.3 of [CLS11]. We have that the corresponding vector bundles are always direct sums of line bundles. We don't want to go into all the details here. You can basically read off the divisors



(a) A fan of type C1 with $\widehat{\Sigma}$ marked in red

(b) A fan of type C2 with the new edges from star-subdivision marked red

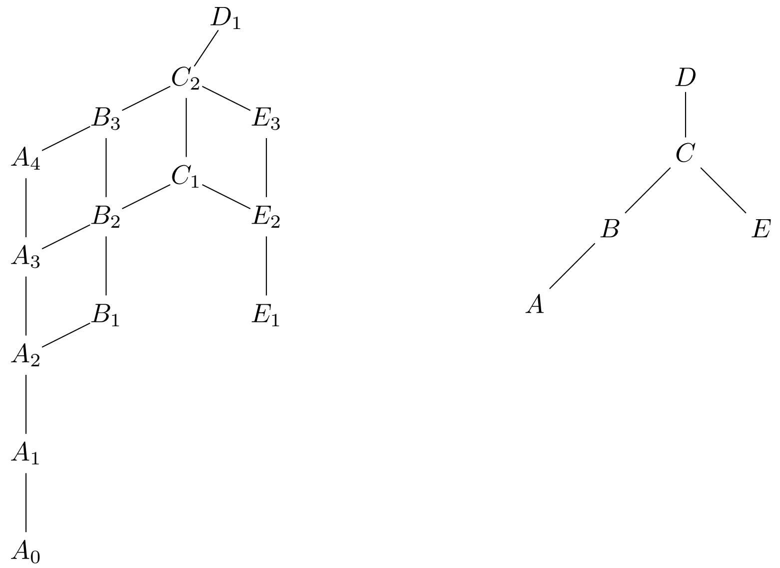
Figure 5.8.: The fans C1 and C2

defining the line bundles from the "heights" of the rays in $\widehat{\Sigma}$ in the direction of the (one-dimensional) fan Σ_0.
In the description below, we always assume that we consider projective bundles over the toric varieties defined by the fans given in Figure 5.5. That is, for example, the fan of $\mathrm{Bl}_1\mathbb{P}^2$ has rays spanned by ν_1, ν_2, ν_3 and ν_4. The divisor on these varieties associated via orbit-cone correspondence to $\mathrm{Cone}(\nu_i)$ is designated by D_i.
Now we can give a list of all chamber types and their corresponding varieties in Figure 5.9.

Type	#	Neighbours	Variety V	V isom. to
A_0	2	$A_1(4)$	$\mathbb{P}^3$	$\mathrm{Bl}_0\mathbb{P}^3$
A_1	8	$A_0(1)$, $A_3(3)$	$\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-D_3))$	$\mathrm{Bl}_1\mathbb{P}^3$
A_2	12	$A_1(2)$, $A_3(2)$, $B_1(1)$	$\mathrm{Bl}_1\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-D_3))$	$\mathrm{Bl}_2\mathbb{P}^3$
A_3	8	$A_2(3)$, $A_4(1)$, $B_2(3)$	$\mathrm{Bl}_2\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-D_3))$	$\mathrm{Bl}_3\mathbb{P}^3$
A_4	2	$A_3(4)$, $B_3(6)$	$\mathrm{Bl}_3\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-D_3))$	$\mathrm{Bl}_4\mathbb{P}^3$
B_1	12	$A_2(1)$, $B_2(2)$	$\mathbb{P}(\mathcal{O}_{\mathrm{Bl}_1\mathbb{P}^2} \oplus \mathcal{O}_{\mathrm{Bl}_1\mathbb{P}^2}(-D_3))$	
B_2	24	$A_3(1)$, $B_1(1)$, $B_3(1)$, $C_1(2)$	$\mathrm{Bl}_1\mathbb{P}(\mathcal{O}_{\mathrm{Bl}_1\mathbb{P}^2} \oplus \mathcal{O}_{\mathrm{Bl}_1\mathbb{P}^2}(-D_3))$	
B_3	12	$A_4(1)$, $B_2(2)$, $C_2(4)$	$\mathrm{Bl}_2\mathbb{P}(\mathcal{O}_{\mathrm{Bl}_1\mathbb{P}^2} \oplus \mathcal{O}_{\mathrm{Bl}_1\mathbb{P}^2}(-D_3))$	
C_1	24	$B_2(2)$, $C_2(1)$, $E_2(1)$	$\mathbb{P}(\mathcal{O}_{\mathrm{Bl}_2\mathbb{P}^2} \oplus \mathcal{O}_{\mathrm{Bl}_2\mathbb{P}^2}(-D_3))$	
C_2	24	$B_3(2)$, $C_1(1)$, $D_1(1)$, $E_3(1)$	$\mathrm{Bl}_1\mathbb{P}(\mathcal{O}_{\mathrm{Bl}_2\mathbb{P}^2} \oplus \mathcal{O}_{\mathrm{Bl}_2\mathbb{P}^2}(-D_3))$	
D_1	4	$C_2(6)$	$\mathbb{P}(\mathcal{O}_{\mathrm{Bl}_3\mathbb{P}^2} \oplus \mathcal{O}_{\mathrm{Bl}_3\mathbb{P}^2}(D_6 - D_3))$	
E_1	4	$E_1(2)$	$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1})$	$(\mathbb{P}^1)^3$
E_2	8	$C_1(3)$, $E_1(1)$, $E_3(1)$	$\mathrm{Bl}_1\mathbb{P}(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1})$	$\mathrm{Bl}_1((\mathbb{P}^1)^3)$
E_3	4	$C_2(6)$, $E_2(2)$	$\mathrm{Bl}_2\mathbb{P}(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1})$	$\mathrm{Bl}_2((\mathbb{P}^1)^3)$
	148			

Figure 5.9.: The chamber types of the secondary fan for $\mathrm{Bl}_4\mathbb{P}^3$

Finally we want to gain a better understanding of the graph G_ν of regular triangulations of $\boldsymbol{\nu}$. As this graph has 148 vertices it would not be feasible to give a picture of the full graph. Instead we will give a contraction of it, where we identify all chambers giving the same isomorphism type of fans Σ. This graph will have 14 vertices corresponding to the 14 types of chambers listed above, and two vertices are connected iff there are two chambers of the corresponding types sharing a common wall. As a consequence we can still distinguish between edges corresponding to star-subdivisions and those coming from flipping walls. The resulting graph is shown in Figure 5.10a. Comparing with the case $n = 2$ pictured in Figure 5.10b one sees that this graph reflects the neighbouring relation of the chambers of types A, B, C, D and E. If Σ'' is obtained by star-subdivision in a fan Σ', both appearing for $\mathrm{Bl}_3\mathbb{P}^2$, then the fan of the $\mathbb{P}^1$-bundle on $X_{\Sigma''}$ is obtained from the fan of the bundle on $X_{\Sigma'}$ by a star-subdivision and a flip.
We want to mention too that the graph in Figure 5.10a not only appears by contract-



(a) $n = 3$

(b) $n = 2$

Figure 5.10.: The graph of chamber types for $\mathrm{Bl}_{n+1}\mathbb{P}^n$

ing G_ν but actually as a subgraph of G_ν. That is you can choose representatives of all 14 classes that are connected exactly in the pattern pictured in Figure 5.10a. In Appendix A we have chosen the representatives of the fan types in this way.

5.3. The general case

5.3.1. Known families of chambers

Proposition 5.5. Let $n \geq 3$. In the secondary fan of $\mathrm{Bl}_{n+1}\mathbb{P}^n$ there are exactly $2\binom{n+1}{k}$ chambers with quotient variety $\mathrm{Bl}_k\mathbb{P}^n$ for $0 \leq k \leq n+1$. Thus this family contributes a total of 2^{n+2} chambers.

Proof. By Proposition 5.1 we see, that it suffices to count the number of possibilities, to build a fan Σ in $N_\mathbb{R}$ isomorphic to $\Sigma_{\mathrm{Bl}_k\mathbb{P}^n}$ such that its rays $\Sigma(1)$ are a subset of $\{\mathrm{Cone}(\nu_i); i = 1, \ldots, 2n+2\}$.
We will first prove the case $k = 0$, i.e. that there are exactly two chamber corresponding to the variety $\mathbb{P}^n$ itself. For this observe that the ray generators of any fan $\Sigma_{\mathbb{P}^n}$ defining the variety $\mathbb{P}^n$ satisfy that no two of them are linearly dependent and that their sum is equal to zero. From this one sees easily that the only possible choices are $\nu_1, \ldots, \nu_{n+1}$ or $\nu_{n+2}, \ldots, \nu_{2n+2}$.
We saw that we obtain the fan of $\mathrm{Bl}_k\mathbb{P}^n$ by k successive star-subdivisions at some of

the $n+1$ vectors ν_i which are not ray generators in the fan $\Sigma_{\mathbb{P}^n}$. We also saw that the order in which we blow them up does not matter. So in the claimed number $2\binom{n+1}{k}$ of chambers for $\mathrm{Bl}_k\mathbb{P}^n$ the factor 2 corresponds to the two different fans for $\mathbb{P}^n$ we found and the factor $\binom{n+1}{k}$ corresponds to the choice of k points at which to star-subdivide. Note that any fan Σ defining the variety $\mathrm{Bl}_{n+1}\mathbb{P}^n$ and using only the rays generated by the ν_i can be obtained from a fan using these rays and defining $\mathbb{P}^n$ by performing star-subdivisions. Together with our result for $k=0$ his implies that there are at most $2\binom{n+1}{k}$ chambers with quotient $\mathrm{Bl}_k\mathbb{P}^n$.
To show the other inequality it suffices to show, that in the fan $\Sigma_{\mathrm{Bl}_k\mathbb{P}^n}$ we can intrinsically identify the rays coming from the fan $\Sigma_{\mathbb{P}^n}$ and those, at which we star-subdivided this fan. For any fan Σ in $N_{\mathbb{R}}$ and a vector $v \in N_{\mathbb{R}}$ we define

$$c(v,\Sigma) = |\{C \in \Sigma \text{ a chamber}; v \in C\}|. \tag{5.1}$$

Now we start with the fan $\Sigma_{\mathbb{P}^n}$ where the rays are generated by $\nu_1,\dots,\nu_{n+1}$. Note that for any ν_i with $i=1,\dots,n+1$ we have $c(\nu_i,\Sigma_{\mathbb{P}^n}) = n$. We want to finish the proof first in case $k=1$, so we star-subdivide one of the chambers. Without loss of generality we choose the chamber $\mathrm{Cone}(e_1,\dots,e_n)$, which is star-subdivided at $\nu_{2n+2} = e_1+\dots+e_n$. We see that $c(\nu_i,\Sigma_{\mathbb{P}^n})$ increases by $n-2\geq 1$ for $i=1,\dots,n$ and $c(\nu_{n+1},\Sigma) = c(\nu_{2n+2},\Sigma) = n$. Thus for $k=1$ we can uniquely identify the ray at which the fan for $\mathbb{P}^n$ was blown up, as it is generated by the sum of all ray generators ν_j with $c(\nu_j,\Sigma) = 2n-2$.
For $k\geq 2$ we star-subdivide the fan from above further at $k-1$ of the vectors $\nu_{n+2},\dots,\nu_{2n+1}$ and obtain a fan Σ. This will automatically increase $c(\nu_{n+1},\Sigma)$ by at least $n-2$ and the value of $c(\nu_i,\Sigma)$, $i=1,\dots,n$, cannot decrease in the process. Conversely, for any ν_j at which we star-subdivided, we will always have $c(\nu_j,\Sigma)=n$. This shows that for a ray $\mathrm{Cone}(\nu_j)\in\Sigma(1)$ we have

$$j\in\{1,\dots,n\} \iff c(\nu_j,\Sigma) > n.$$

Hence as claimed we can intrinsically identify the rays in $\Sigma_{\mathrm{Bl}_k\mathbb{P}^n}$ which came from the fan $\Sigma_{\mathbb{P}^n}$ we started with. This finishes the proof. □

While Proposition 5.1 implies (with a short argument) that all the varieties $\mathrm{Bl}_k\mathbb{P}^n$ will occur as quotients corresponding to chambers of the secondary fan, we have seen that even in the case $n=2$ there are other chambers. In fact, for $n=2,3$ we observed that all chambers corresponded either to $\mathbb{P}^n$ itself or to the blow-up of some points on a projective bundle over a variety, that appeared in the case $n-1$. For $n=3$ we explicitly mentioned it and in case $n=2$ this is even easier to verify. For an argument that $\mathrm{Bl}_1\mathbb{P}^2$ is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ see Section 2.3.3 and $\mathbb{P}^1\times\mathbb{P}^1$ is naturally a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$.
Now it is easy to show that this "recursive" family of varieties generalizes to all $n\geq 4$. Let Σ' in $N_{\mathbb{R}}$ be a fan corresponding to a chamber for $\mathrm{Bl}_n\mathbb{P}^{n-1}$. We know that its rays will be generated by some of the vectors $\nu'_1,\dots,\nu'_{2n+2}\in N_{\mathbb{R}}$. Here we write ν'_i instead of ν_i to distinguish them from the corresponding vectors in $N_{\mathbb{R}}\oplus\mathbb{R}$. Then we can build a fan $\Sigma_{\Sigma',n+1}$ in $N_{\mathbb{R}}\oplus\mathbb{R} = (N\oplus\mathbb{Z})_{\mathbb{R}}$ using the vectors $\nu_1,\dots,\nu_{2(n+1)+2}$ as follows.

First we associate the vectors ν'_i with the vectors ν_j which are not e_{n+1} or $-e_{n+1}$ in the following pattern

$$\nu'_1 \leftrightarrow \nu_1, \dots, \nu'_n \leftrightarrow \nu_n, \nu'_{n+1} \leftrightarrow \nu_{n+2}, \nu'_{n+2} \leftrightarrow \nu_{n+3}, \dots, \nu'_{2n+1} \leftrightarrow \nu_{2n+2}, \nu'_{2n+2} \leftrightarrow \nu_{2n+4}.$$

One sees easily that this ensures that under the projection $\pi : (N \oplus \mathbb{Z})_{\mathbb{R}} \to N_{\mathbb{R}}$ any vector ν'_i is the image of its associated vector ν_j and the kernel is spanned by e_{n+1}. Now we form a (codimension 1) fan $\widehat{\Sigma}$ in $(N \oplus \mathbb{Z})_{\mathbb{R}}$, where for every cone $\sigma \in \Sigma'$ spanned by some of the vectors ν'_i we define the corresponding cone $\widehat{\sigma} \subset (N \oplus \mathbb{Z})_{\mathbb{R}}$ to be spanned by their associated vectors ν_j. We define

$$\widehat{\Sigma} = \{\widehat{\sigma}; \sigma \in \Sigma'\}.$$
$$\Sigma_{\Sigma',n+1} = \{\widehat{\sigma} + \mathrm{Cone}(e_{n+1}); \sigma \in \Sigma'\} \cup \widehat{\Sigma} \cup \{\widehat{\sigma} + \mathrm{Cone}(-e_{n+1}); \sigma \in \Sigma'\}.$$

Then $\Sigma_{\Sigma',n+1}$ is a simplicial fan and by construction it is immediate that $\Sigma_{\Sigma',n+1}$ is split by Σ' and $\Sigma_0 = \{\mathrm{Cone}(e_{n+1}), \{0\}, \mathrm{Cone}(-e_{n+1})\}$ under the map π. Thus we have that its toric variety is a $\mathbb{P}^1$-fibre bundle over $X_{\Sigma'}$. Using the description in Proposition 7.3.3 of [CLS11], one sees easily, that it is actually a projective bundle over $X_{\Sigma'}$ and hence itself projective. Thus by Proposition 5.1 all fans constructed like this correspond to chambers in the secondary fan for $\mathrm{Bl}_{n+2}\mathbb{P}^{n+1}$.
Now it is easy to see that the vector e_{n+1} is not special in any regard. Actually we can construct a fan $\Sigma_{\Sigma',i}$ where Σ_0 is any of the $n+2$ fans $\{\mathrm{Cone}(\nu_i), \{0\}, \mathrm{Cone}(-\nu_i)\}$ for $i = 1, \dots, n+2$. For $i = 1, \dots, n+1$ it is fairly obvious how to identify the sets of ray generators

$$\{\nu'_k; k = 1, \dots, 2n+2\} \text{ and } \{\nu_j; j = 1, \dots, 2n+4\} \setminus \{e_i, -e_i\}.$$

For $i = n+2$ this will involve some choice of isomorphism $\mathbb{Z}^{n+1}/\mathbb{Z}(1, \dots, 1) \cong \mathbb{Z}^n$, but this we can fix by demanding that $e_j \mapsto e_j$ for $j = 1, \dots, n$, forcing $e_{n+1} \mapsto -e_1 - e_2 \dots - e_n$.
We now recursively define a family $\mathcal{F}_n$ of fans. Let $\mathcal{F}_2$ be the set of fans Σ in $(\mathbb{Z}^2)_{\mathbb{R}}$ corresponding to a chamber of the secondary fan for $\mathrm{Bl}_3\mathbb{P}^2$. If we have defined $\mathcal{F}_n$ for $n \geq 2$ we set $\mathcal{F}_{n+1}$ to be the set of fans in $(\mathbb{Z}^{n+1})_{\mathbb{R}}$ of one of the following forms:

- $\Sigma_{\mathbb{P}^{n+1}}$ and $-\Sigma_{\mathbb{P}^{n+1}}$ with ray generators $\nu_1, \dots, \nu_{n+2}$ and $\nu_{n+3}, \dots, \nu_{2n+4}$, respectively
- $\Sigma_{\Sigma',i}$ for $\Sigma' \in \mathcal{F}_n$ and $i = 1, \dots, n+2$
- all star subdivisions of fans Σ of the above two types at some of the ray generators ν_i not used in Σ.

Then by the discussion above together with Proposition 4.6 and the fact that the vector configurations ν_i are geometric, $\mathcal{F}_n$ will correspond to a subset of the set of chambers for the secondary fan for $\mathrm{Bl}_{n+1}\mathbb{P}^n$. Note firstly that at least in the case $n = 3$ all chambers correspond to fans in $\mathcal{F}_3$. Secondly one sees that in general the fans of the product $\mathbb{P}^1 \times \mathbb{P}^1 \times \dots \times \mathbb{P}^1$ of n copies of $\mathbb{P}^1$ is contained in $\mathcal{F}_n$.
We now derive the asymptotic of the number of fans in the family $\mathcal{F}_n$.

Theorem 5.6. The function $n \mapsto |\mathcal{F}_n|$ is in $\exp(\theta(n \log n))$.

Proof. To find the asymptotic behaviour it suffices to show some bounds for the case $n \geq 3$. Note that in this case every fan $\Sigma \in \mathcal{F}_n$ not isomorphic to $\Sigma_{\mathbb{P}^n}$ arises, modulo automorphisms of the vector configuration $\boldsymbol{\nu}$ in the following way:

1. We start with a fan Σ_k corresponding to varieties $\mathbb{P}^k$, $k = 3, \ldots, n-1$ or with one of the fans Σ_k in $\mathcal{F}_2$ (in which case we set $k = 2$) defined on the vectors $\nu_1, \ldots, \nu_{2k+2}$. Note that in the first case $k+1$ of them are not used as ray generators. The indices of unused vectors are stored in $\mathcal{I} = \mathcal{I}_k$. Set $i = k$.

2. We construct a new fan Σ_{i+1} corresponding to the $\mathbb{P}^1$-bundle over X_{Σ_i}, where the vectors $e_{i+1}, -e_{i+1}$ generate the $\mathbb{P}^1$-subfan in Σ_{i+1}. We increase i by one.

3. Star-subdivisions at some of the yet unused vectors corresponding to indices $I_i \subset \mathcal{I}_i$ are performed. Remove I_i from $\mathcal{I}_i$.

4. If $i = n$ we get our resulting fan $\Sigma = \Sigma_n$. Otherwise return to Step 2.

Note that the resulting fan is uniquely determined by the information k, Σ_k and for every $j \in \mathcal{I}_k$ the information, at which of the $n-k$ possible occurences of Step 3 they were blown up (or not at all), amounting to $n-k+1$ possibilities.
For a lower bound on $|\mathcal{F}_n|$ we ignore the case $k = 2$, i.e. choosing as starting fan one of those in $\mathcal{F}_2$. Hence for every $k = 3, \ldots, n$ we have two possibilities to chose a fan $\Sigma_k \cong \Sigma_{\mathbb{P}^k}$ and $(n-k+1)^{k+1}$ choices when (or if) to blow up the $k+1$ initial unused ray generators. We remark that all fans obtained this way are pairwise distinct, as for a fan Σ split by Σ' and Σ_0 we can reconstruct Σ' knowing Σ_0. This gives a lower bound

$$|\mathcal{F}_n| \geq 2 \sum_{k=3}^{n-1} (n-k+1)^{k+1}. \tag{5.2}$$

Restricting ourself to the case $n \geq 6$ we take the term for $k = \lfloor n/2 \rfloor$ and obtain

$$|\mathcal{F}_n| \geq 2\left(n - \left\lfloor \frac{n}{2} \right\rfloor + 1\right)^{\lfloor \frac{n}{2} \rfloor + 1} \geq \left(\frac{n}{2}\right)^{\frac{n}{2}} = \exp((\log n - \log 2)\frac{n}{2}) \in \exp(\theta(n \log n)).$$

On the other hand to obtain an upper bound on $|\mathcal{F}_n|$ we first bound the number of chambers obtained by the algorithm above. For this we certainly have the right side of inequality (5.2) counting the cases $k \geq 3$ and we have extra terms of the form $(n-1)^r$ corresponding to the varieties where we chose $k = 2$, $\Sigma_2 \in \mathcal{F}_2$ and have r unused ray generators. Those terms as well as the summand 2 corresponding to $\mathbb{P}^n$ itself are all in $O(n^2)$. To obtain a bound for all the fans in $\mathcal{F}_n$ we multiply with the size $2(n+1)!$ of the full symmetry group of $\boldsymbol{\nu}$ and obtain

$$|\mathcal{F}_n| \leq 2(n+1)! \left(O(n^2) + \sum_{k=3}^{n-1} (n-k+1)^{k+1} \right) \tag{5.3}$$

Now by Stirling's approximation the term $2(n+1)!$ is in $\exp(\theta(n \log n))$. To bound the other term we can certainly neglect the term $O(n^2)$ and we see

$$\sum_{k=2}^{n-1}(n-k+1)^{k+1} \le \sum_{k=3}^{n-1} n^{k+1} \le \sum_{k=0}^{n} n^k = \frac{n^{n+1}-1}{n-1} \in \exp(\theta(n \log n)).$$

Hence we have proved the desired upper and lower bounds. □

As a last result in this section we also want to say something about the asymptotic behaviour of the number of chambers in the secondary fan for $\mathrm{Bl}_{n+1}\mathbb{P}^n$. In the last sections we computed this number in the cases $n = 1, 2, 3$. To compute the secondary fan for bigger n we used the software `polymake` (see [GJ00]) as an interface to the `gfan`-package (see [Jen]). This package uses Gröbner bases to compute the secondary fan. Using it we were able to also find the number of chambers in the case $n = 4$.

n	number of chambers
1	4
2	18
3	148
4	3285

By going more carefully and case-by-case through the proof of Theorem 5.6 one can show, that the number $|\mathcal{F}_4|$ is strictly less than the total number of chambers in the secondary fan of $\mathrm{Bl}_5\mathbb{P}^4$. Thus, unfortunately, our family $\mathcal{F}_n$ does not cover all the chambers.
Nonetheless we want to give an upper bound on the number of chambers, for which we will use the following Lemma.

Lemma (see for instance [Zas75])**.** The number of full-dimensional regions defined by a linear arrangement of N hyperplanes in dimension D ($N \ge D$) is bounded above by

$$2\sum_{i=0}^{D-1}\binom{N-1}{i}.$$

Corollary 5.7. The number of chambers in the secondary fan for $\mathrm{Bl}_{n+1}\mathbb{P}^n$ is in $\exp(\Omega(n \log n))$ and in $\exp(O(n^2))$.

Proof. The lower bound of $\exp(\Omega(n \log n))$ follows from the proof of Theorem 5.6, as it is even a lower bound for $|\mathcal{F}_n|$. As for the upper bound, we have seen in Lemma 5.3 that the codimension-1 cones spanned by subsets of $\boldsymbol{\beta}$ correspond to the 2^{n+1} cocircuits of the form $\mathrm{U}(S)$ and the $n+1$ cocircuits $u(i)$. The interiors of chambers of the secondary fan are the connected components of C_β with the union of these cones removed. Then one easily sees that this number can only increase when taking the full $\mathbb{R}^{n+2}$ and removing all $2^{n+1}+n+1$ hyperplanes spanned by these cones. Using the

Lemma above we get an upper bound of

$$2\sum_{i=0}^{n+1}\binom{2^{n+1}+n}{i} \le 2(n+2)\binom{2^{n+1}+n}{n+1} \le 2(n+2)(2^{n+1}+n)^{n+1} \in \exp(O(n^2))$$

□

We want to remark that from the proofs above one can extract more explicit bounds and with more work one can also significantly improve the constants in those bounds compared to what can be extracted from the proofs given here. However we wanted to avoid giving very long and technical proofs and we felt that the asymptotics presented here give a good impression of the scale at which the number of chambers grows.

5.3.2. $\mathrm{Bl}_k\mathbb{P}^n$ for $k \le n+1$

We also want to give an intuition why understanding the secondary fan for $\mathrm{Bl}_{n+1}\mathbb{P}^n$ for all n also enables us to understand the corresponding secondary fans for blow-ups of less than $n+1$ points on $\mathbb{P}^n$. Observe that all varieties $\mathrm{Bl}_k\mathbb{P}^n$ are projective and we can build their fan using rays generated by $\tilde{\boldsymbol{\nu}} = \{\nu_1, \dots, \nu_k, \nu_{n+2}, \dots, \nu_{2n+2}\}$. Here the last $n+1$ rays define the (second version of) a fan for $\mathbb{P}^n$ and the first k vectors are introduced by star-subdivisions. Then Proposition 5.1 tells us that we get a quotient description of $\mathrm{Bl}_k\mathbb{P}^n$ from the vector configuration $\tilde{\boldsymbol{\nu}}$ and its Gale dual configuration $\tilde{\boldsymbol{\beta}}$. By what we have discussed in Section 2.5.3 concerning the dual operations of deletion and contraction on oriented matroids, we see that the configuration $\tilde{\boldsymbol{\beta}}$ will be an iterated contraction of the original vectors β_i. We have the following result.

Theorem 5.8. The vectors $\tilde{\boldsymbol{\beta}} \subset \mathbb{R}^{k+1}$ that are the Gale dual configuration of $\tilde{\boldsymbol{\nu}}$ can be chosen as $\pi(\boldsymbol{\beta} \setminus \{\beta_{k+1}, \dots, \beta_{n+1}\})$ where $\pi : \mathbb{R}^{n+2} \to \mathbb{R}^{k+1}$ is the coordinate projection sending $e_i \mapsto e_i$ for $i = 1, \dots, k$, $e_i \mapsto 0$ for $i = k+1, \dots, n+1$ and $e_{n+2} \mapsto e_{k+1}$. One sees that $\tilde{\boldsymbol{\beta}}$ is the union of the set $\boldsymbol{\beta}'$ obtained for $\mathrm{Bl}_k\mathbb{P}^{k-1}$ together with $n+1-k$ times the vector e_{k+1}. Correspondingly, the secondary fan for $\mathrm{Bl}_k\mathbb{P}^n$ is the union of the secondary fan of $\mathrm{Bl}_k\mathbb{P}^{k-1}$ with the cone $\sigma = \mathrm{Cone}(e_{k+1}, e_1 + e_{k+1}, \dots, e_k + e_{k+1})$.
Moreover, there is a canonical bijection between the chambers of the secondary fan for $\mathrm{Bl}_k\mathbb{P}^n$ and those chambers in the secondary fan of $\mathrm{Bl}_{n+1}\mathbb{P}^n$ containing $\beta_{k+1}, \dots, \beta_{n+1}$. This bijection also identifies the corresponding fans Σ and hence the associated quotient varieties X_Σ.

Proof. By the results in Section 2.5.3 we obtain $\tilde{\boldsymbol{\beta}}$ by sucessively contracting with $\beta_{k+1}, \dots, \beta_{n+1}$. But this is easily seen to correspond to the linear map π defined above. The result about the explicit form of the vectors in $\tilde{\boldsymbol{\beta}}$ is immediate from the description of $\boldsymbol{\beta}$ in Section 5.1.2. One sees that the cone $C_{\tilde{\beta}}$ spanned by $\tilde{\boldsymbol{\beta}}$ is the union of the support of the original secondary fan for $\mathrm{Bl}_k\mathbb{P}^{k-1}$ and the cone σ above. It is also easy to see that all codimension 1 cones spanned by subsets of $\tilde{\boldsymbol{\beta}}$ containing the vector e_{k+1} are contained in facets of $C_{\tilde{\beta}}$. Together with the construction of the secondary fan given in Section 3.4.3, this implies that σ is a chamber of the secondary fan and hence this fan has the form claimed above.

As for the claimed bijection describing the quotients that arise in the secondary fan of $\mathrm{Bl}_k\mathbb{P}^n$ it follows from Proposition 5.1 that these correspond to chambers in the secondary fan of $\mathrm{Bl}_{n+1}\mathbb{P}^n$ with $I_\varnothing \supset \{k+1, \dots, n+1\}$ and Proposition 4.3 tells us that this last condition is equivalent to the chambers containing $\beta_{k+1}, \dots, \beta_{n+1}$. □

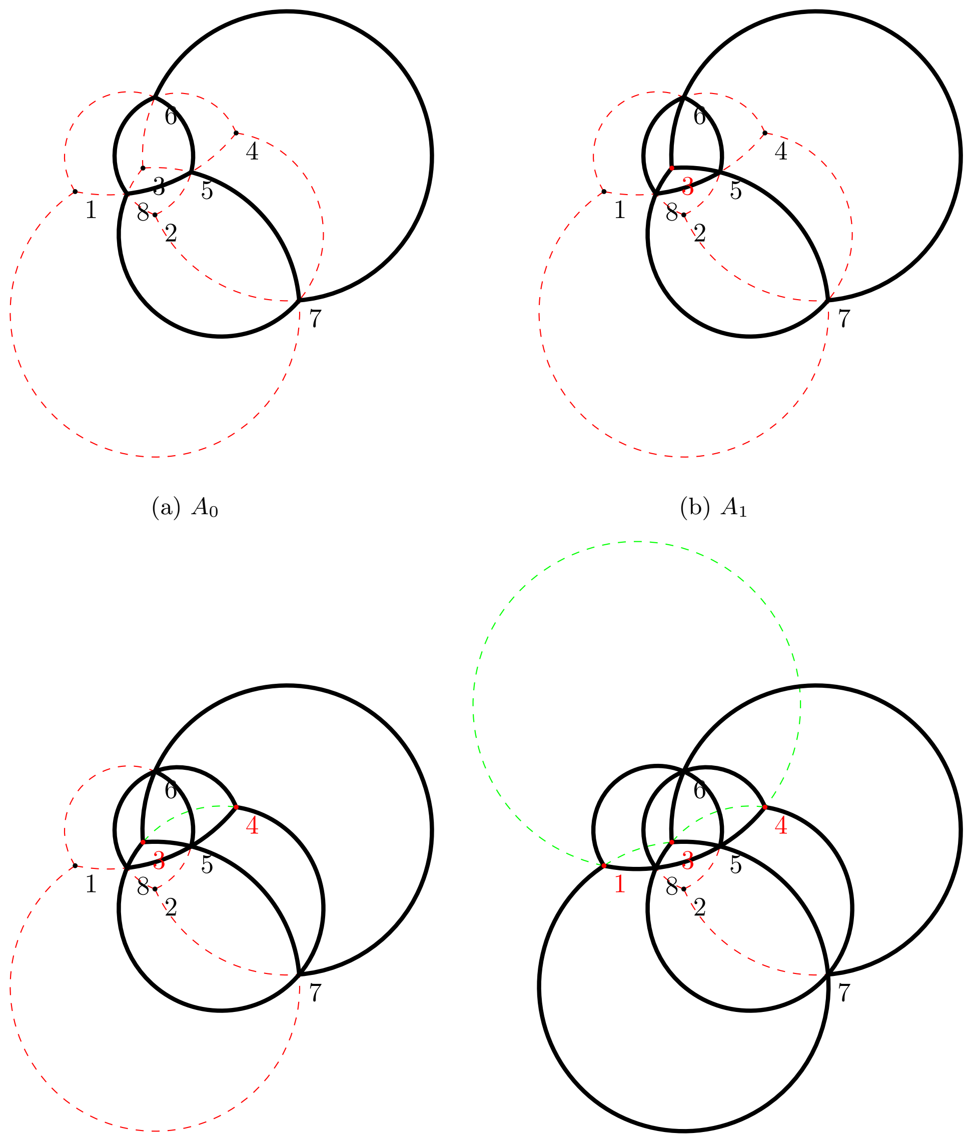
In the Theorem above we approach the secondary fan for $\mathrm{Bl}_k\mathbb{P}^n$ on the one hand using the secondary fan for $\mathrm{Bl}_k\mathbb{P}^{k-1}$ and on the other hand by looking at chambers in the secondary fan of $\mathrm{Bl}_{n+1}\mathbb{P}^n$. It would be interesting to describe a connection between those two approaches.

A. A gallery of all chamber types for $\mathrm{Bl}_4\mathbb{P}^3$

In Section 5.2.3 we looked at a quotient description for $\mathrm{Bl}_4\mathbb{P}^3$, the corresponding vector configurations $\boldsymbol{\nu} \subset \mathbb{R}^3$ and $\boldsymbol{\beta} \subset \mathbb{R}^5$ and at the secondary fan Σ_{GKZ} in $\mathbb{R}^5$.
By using the `ccr` (coarsest common refinement) function in the `Polyhedra`-package for `Macaulay2` (see [GS]) we could identify all 148 chambers of the secondary fan. For every chamber $\Gamma_{\Sigma,I_\varnothing}$ we obtained the corresponding fan Σ in $\mathbb{R}^3$. This was done by chosing a point $b \in \mathrm{Int}(\Gamma_{\Sigma,I_\varnothing})$, taking a preimage $a \in \mathbb{R}^8$, computing the corresponding polytope P_a and taking its normal fan.
Now however we are faced with the task of identifying the corresponding toric variety X_Σ. Although a computer can algorithmically determine if two fans are isomorphic, it would need a database of examples and constructions to compare the given fan to. Thus the natural choice is to identify the varieties by hand. For this purpose, we have to create some (mental) picture of the fan Σ. As it lives in three dimensional space, this is already a non-trivial task. However, the cones in Σ are of course invariant under scaling with a positive constant. Hence all the information we need is contained in the intersection of the cones $\sigma \in \Sigma$ with the sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. The (nonempty) intersections that occur are points, arcs on great circles and spherical triangles (as the fans Σ are simplicial).
To finally obtain a flat, two-dimensional picture, we choose some sufficiently generic point $p \in \mathbb{S}^2$ and apply stereographic projection from p to the tangent space $T_{-p}\mathbb{S}^2$. One can verify that under this transformation, great circles are mapped to circles in the plane. The geodesic triangles are correspondingly mapped to circular triangles. The only exception is the triangle in which p is contained. It is mapped to the "outer face" in the planar picture, i.e. its interior maps to the unbounded component of the complement of the union of all circular arcs.
Concerning the implementation, we extracted the combinatorial information of the fans Σ from `Macaulay2`. In our case this means we exported all pairs $\{i,j\}$ of indices $1 \le i,j \le 8$ with $\mathrm{Cone}(\nu_i,\nu_j) \in \Sigma$. Then we read those into a `C++` program. Using the `SPHERE_STEREOGRAPH` package by John Burkardt (see [Bur]) we computed the images $\bar{\nu}_i$ of the normalized vectors $\nu_i/|\nu_i| \in \mathbb{S}^2$ under the stereographic projection from the point $(0.8,-0.8,-1.8)/|(0.8,-0.8,1.8)|$. For any pair $\{i,j\}$ we used the projection of the additional point $(\nu_i+\nu_j)/|\nu_i+\nu_j|$ to uniquely determine the desired circle through $\bar{\nu}_i, \bar{\nu}_j$. Then, according to the combinatorial information of the fan Σ, we created `TikZ` code drawing the corresponding circular arcs and the points $\bar{\nu}_i$. For the incredibly useful LaTeXpackage `TikZ` and its wonderful "TikZ & PGF manual" see [Tan].
Using another `C++` program we determined the isomorphism classes of the fans Σ. It turned out that there are 14 such classes. Below follow pictures of fans from each of

those classes. The fan itself is printed in black, possible star-subdivisions are marked in dotted red lines. Nodes where a star-subdivision can be reversed are also marked in red and edges that appear in a chamber connected to the current chamber by a flip are shown dotted green. Note that the vectors ν_i are simply denoted by i for brevity.

Figure A.1.: The fans associated to chambers of the secondary fan for $\mathrm{Bl}_4\mathbb{P}^3$

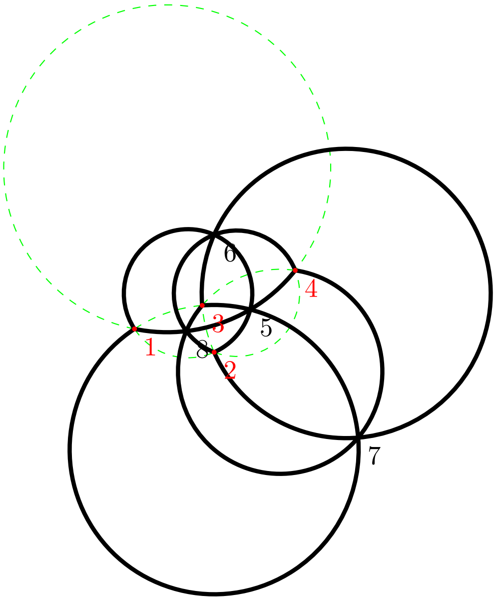


(a) A_0

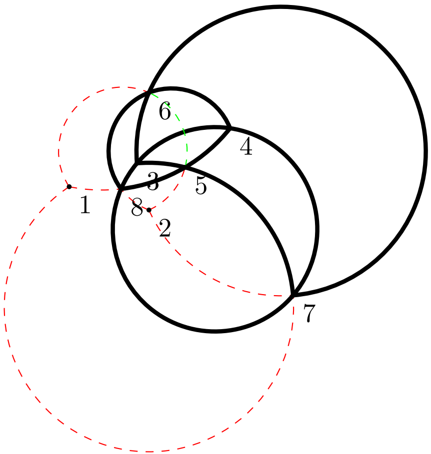
(b) A_1

(c) A_2

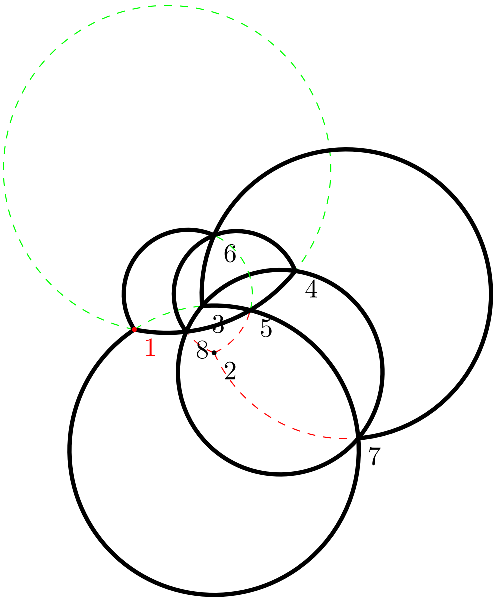
(d) A_3



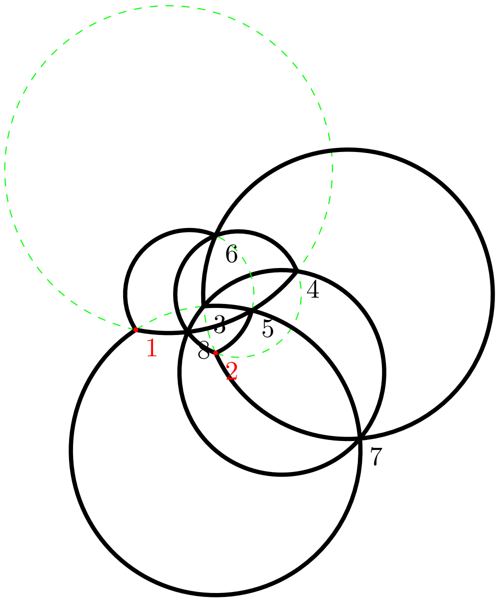
(e) A_4



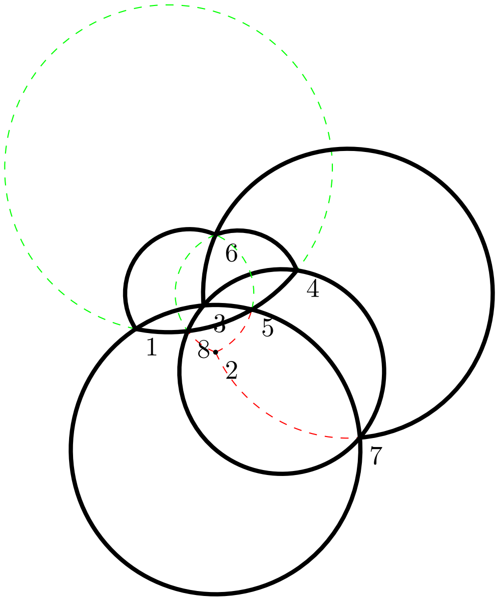
(f) B_1



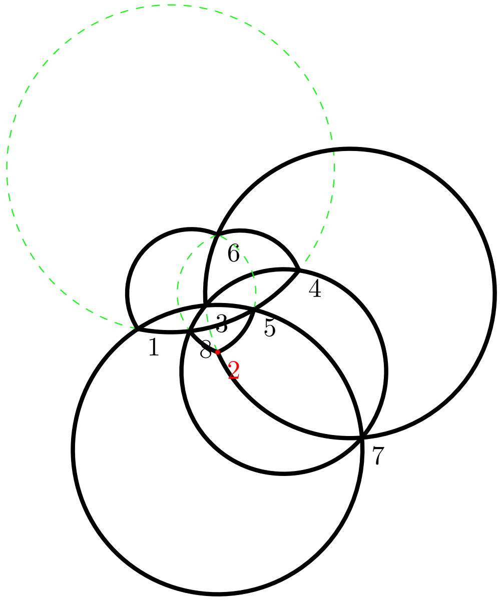
(g) B_2



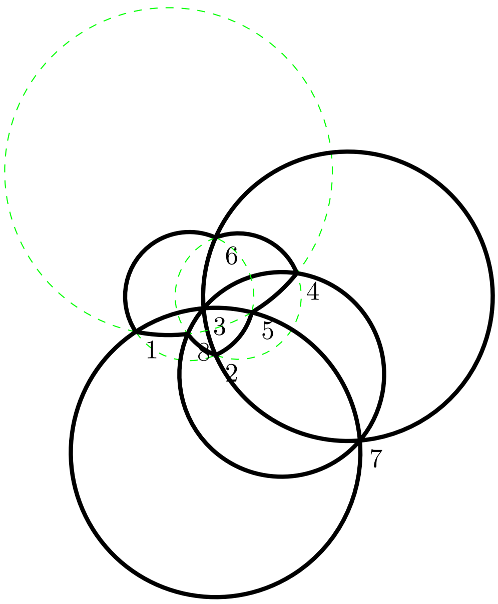
(h) B_3



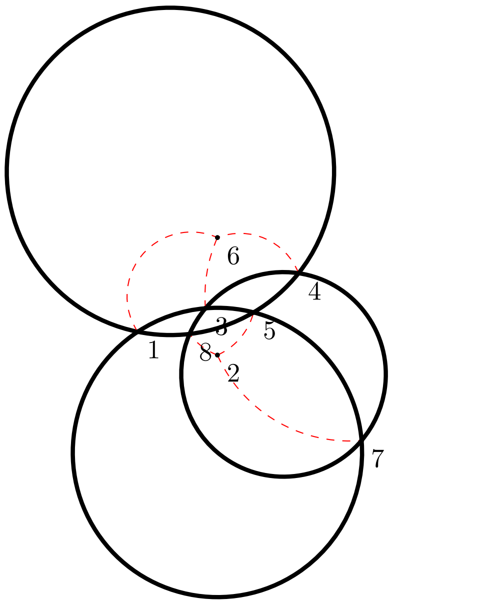
(i) C_1



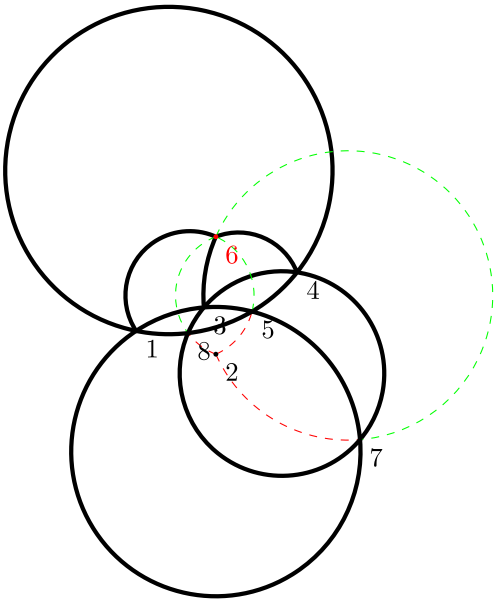
(j) C_2



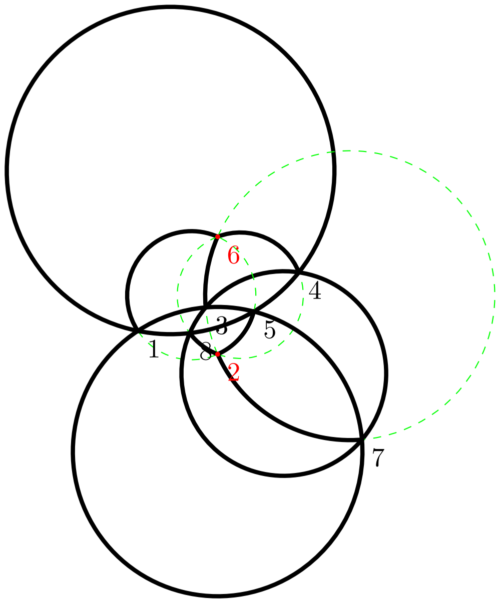
(k) D_1



(l) E_1



(m) E_2



(n) E_3

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