Pandharipande-Yin : Relations in the tautological ring of the moduli of K3 surfaces

§1 The tautological ring of the moduli of K3 surfaces

$$\mathcal{M}_{2\ell} = \left\{ (X,H) \;\middle|\; \begin{array}{l} X \text{ nonsing. proj. K3} \\ H \in \mathrm{Pic}(X) \text{ q.pol.},\ H^2 = 2\ell \end{array} \right\}$$ moduli of quasi-polarized K3 surf.

$\ell \geq 1$

$\Lambda \supset (2\ell)$ lattice

$i_\Lambda : \mathcal{M}_\Lambda \longrightarrow \mathcal{M}_{2\ell}$ Noether-Lefschetz loci

$\mathcal{M}_\Lambda$: Λ-polarized K3 surfaces

Two subrings of Chow rings $A^*(\mathcal{M}_{2\ell})$: always $\mathbb{Q}$-coeff

- $NL^*(\mathcal{M}_{2\ell}) = \langle (i_\Lambda)_*[\mathcal{M}_\Lambda] \rangle$ subring gen. by NL loci
- For $\pi_\Lambda : \mathcal{X}_\Lambda \to \mathcal{M}_\Lambda$ univ. K3, $L_1, \dots, L_k \in \Lambda$

$$K_{[L_1^{a_1} \cdots L_k^{a_k};\, b]} = (\pi_\Lambda)_* \left(\mathcal{L}_1^{a_1} \cdots \mathcal{L}_k^{a_k} \cdot c_2(T_{\pi_\Lambda})^b \right) \in A^{\sum a_i + 2b - 2}(\mathcal{M}_\Lambda)$$

$$R^*(\mathcal{M}_{2\ell}) = \langle (i_\Lambda)_* (\text{products of } K_{[\dots]}\text{-classes}) \rangle$$ strict tautological ring

Theorem 1 (MOP Conjecture)

$$NL^*(\mathcal{M}_{2\ell}) = R^*(\mathcal{M}_{2\ell})$$

"$\subseteq$" obvious

"$\supseteq$" write K-classes on $\mathcal{M}_\Lambda$ in terms of NL classes

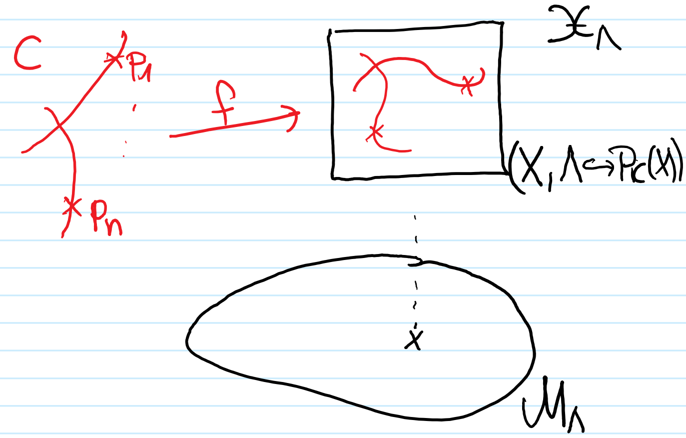
Idea of proof

Use relations between cycles in $\mathcal{X}_\Lambda^\ell \to \mathcal{M}_\Lambda$ proved using the moduli space of stable maps

§2 Moduli spaces of stable maps & the export construction

Let $g,n \geq 0$ integers, $\pi_\Lambda : \mathcal{X}_\Lambda \to \mathcal{M}_\Lambda$, $L \in \Lambda$

$$\overline{\mathcal{M}}_{g,n}(\pi_\Lambda, L) = \{(C,p_1,\dots,p_n) \xrightarrow{f} X\}$$



where

- C connected proj. curve, at worst nodal sing., arithm. genus g
- $p_1,\dots,p_n \in C$ smooth distinct pts
- $(X, \Lambda \hookrightarrow Pic(X)) \in \mathcal{M}_\Lambda$
- $f: C \longrightarrow X$ morphism st. $f_*[C] = c_1(L) \in H_2(X,\mathbb{Z})$
- $\mathrm{Aut}((C,p_1,\dots,p_n) \xrightarrow{f} X)$ finite.

<u>Fact</u> $\overline{\mathcal{M}}_{g,n}(\pi_\Lambda, L)$ is Deligne-Mumford stack, proper over $\mathcal{M}_\Lambda$ with a natural class

$$[\overline{\mathcal{M}}_{g,n}(\pi_\Lambda,L)]^{red} \in A_{\dim(\mathcal{M}_\Lambda)+g+n}(\overline{\mathcal{M}}_{g,n}(\pi_\Lambda,L))$$

reduced virtual fundamental class

Have a diagram

$$\begin{tikzcd}
& ((C,p_1,\dots,p_n) \xrightarrow{f} X) \arrow[dl, mapsto] \arrow[dr, mapsto] & \\
(C,p_1,\dots,p_n) & \overline{\mathcal{M}}_{g,n}(\pi_\Lambda, L) \arrow[dl, "\tau"'] \arrow[dr, "\varepsilon^n"] & (f(p_1),\dots,f(p_n)) \\
\overline{\mathcal{M}}_{g,n} & & \mathcal{X}_\Lambda^n = \mathcal{X}_\Lambda \times_{\mathcal{M}_\Lambda} \cdots \times_{\mathcal{M}_\Lambda} \mathcal{X}_\Lambda
\end{tikzcd}$$

moduli of stable curves $\overline{\mathcal{M}}_{g,n}$

<u>Strategy</u>

- Start with relation $Rel = \sum R_i = 0 \in A^*(\overline{\mathcal{M}}_{g,n})$
- Compute the <u>export</u> via diagram above:

$$\varepsilon^n_*(\tau^* Rel) \cap [\overline{\mathcal{M}}_{g,n}(\pi_\Lambda, L)]^{red} = 0 \in A_*(\mathcal{X}_\Lambda^n)$$

- Profit.

Conn. to Yagna's talk

$$\begin{tikzcd}
Q^{\pi}_{H/X}(\mathbb{C}^2) \arrow[d, "\pi"] & \text{relations from localiz.} \arrow[d, "\pi_*"] \\
\mathcal{M}_{2\ell} & \text{rel. in } \mathcal{M}_{2\ell}
\end{tikzcd}$$

However, before we can state the results, we need a more honest/precise description of universal classes on $\mathcal{X}_\Lambda$.

§3 The $\mathcal{L}$ and K-classes revisited & the taut. ring of $\mathcal{X}_\Lambda^n$

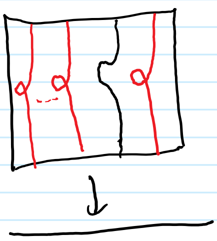
Copied from above:

For $\pi_\Lambda : \mathcal{X}_\Lambda \to \mathcal{M}_\Lambda$ univ. K3, $L_1, \ldots, L_k \in \Lambda$

$$K_{[L_1^{a_1} \cdots L_k^{a_k}; b]} = (\pi_\Lambda)_* \left(\mathcal{L}_1^{a_1} \cdots \mathcal{L}_k^{a_k} \cdot c_2(T_{\pi_\Lambda})^b \right) \in A^{\sum a_i + 2b - 2}(\mathcal{M}_\Lambda)$$

$\mathcal{L}_i \in \mathrm{Pic}(\mathcal{X}_\Lambda)$ only def. upto pullbacks from $\mathrm{Pic}(\mathcal{M}_\Lambda)$

Idea Consider ellipt. fibred K3 surfaces



24 rat'l fibres R_i

$$\Lambda = \begin{pmatrix} 2\ell & 1 \\ 1 & 0 \end{pmatrix} \ni F \text{ fiber class}$$

$$\Rightarrow F = \frac{1}{24} \cdot \sum_{i=1}^{24} [R_i] \in A^1(X)$$

To do this universally over $\mathcal{M}_\Lambda$: use stable maps!

Caveat: Need to restrict to $L \in \Lambda$ admissible, i.e.

(i) $L = m \cdot \tilde{L}$ w/ $\tilde{L}$ primitive, $m > 0$, $\tilde{L}^2 \geq -2$

(ii) $H \cdot L \geq 0$

and in case of equality in (ii)

(ii') L is effective.

For such $L \in \Lambda$ we define

$$\mathcal{L} = \frac{1}{N_0(L)} \cdot \varepsilon_* [\overline{M}_{0,1}(\pi_\Lambda, L)]^{\mathrm{red}} \in A^1(\mathcal{X}_\Lambda, \mathbb{Q})$$

for $N_0(L)$ defined by

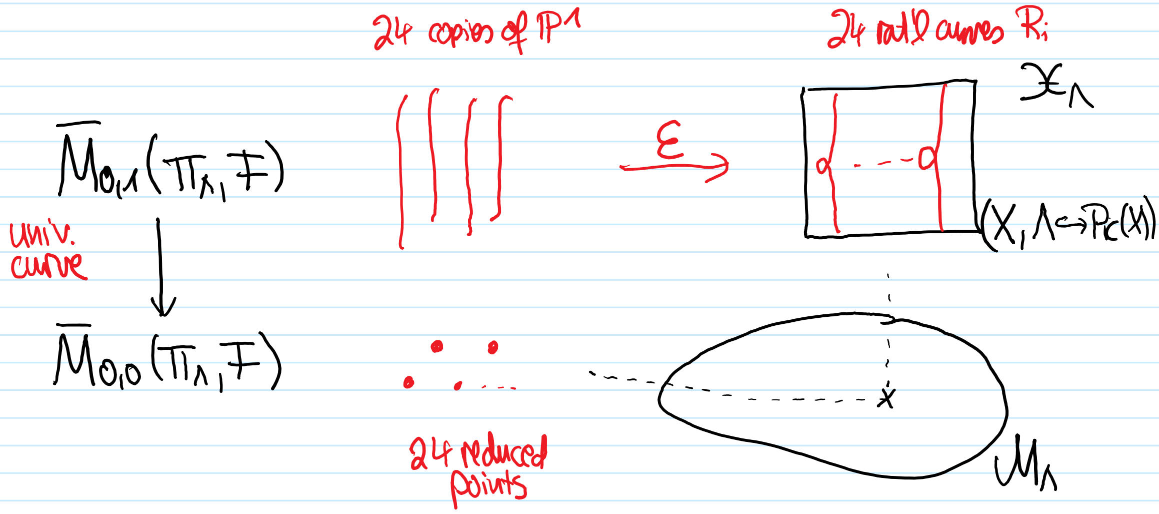
$$(\varepsilon_0)_* [\overline{M}_{0,0}(\pi_\Lambda, L)]^{\mathrm{red}} = N_0(L) \cdot [\mathcal{M}_\Lambda]$$

$$\begin{array}{c} \overline{M}_{0,0}(\pi_\Lambda, L) \\ \downarrow \varepsilon^0 \\ \mathcal{M}_\Lambda \end{array}$$

$\Leftrightarrow N_0(L) = \int_{[\overline{M}_{0,0}(X,L)]^{red}} 1$

~~$\circ$~~ later $\quad = \text{virtual/reduced \# rat'l curves in } |L|$

<u>Example</u>



<u>Summary</u> $L \in \Lambda$ admissible $\Rightarrow \mathcal{L} \in A^1(\mathcal{X}_\Lambda, \mathbb{Q})$

$\rightsquigarrow K_{[L_1^{a_1} \cdots L_k^{a_k}; b]}$ for $L_1, \ldots, L_k$ admissible.

<u>Def</u> The strict tautological ring $R^*(\mathcal{X}_\Lambda^n) \subseteq A^*(\mathcal{X}_\Lambda, \mathbb{Q})$ is defined as the subring gen. by pushforwards

$$\mathcal{X}_{\Lambda'}^n \longrightarrow \mathcal{X}_\Lambda^n$$

from Noether-Lefschetz subloci of all products of

$\to$ $\pi_{\Lambda'}^n$-relative diagonals $\Delta_{(i_1, \ldots, i_l)} \subseteq \mathcal{X}_{\Lambda'}^n$

$\to$ pull-backs $\left\{ \begin{array}{l} \mathcal{L}_{(i)} \text{ of } \mathcal{L} \in A^1(\mathcal{X}_{\Lambda'}) \\ c_2(\mathcal{T}_{\pi_{\Lambda'}(i)}) \text{ of } c_2(\mathcal{T}_{\pi_{\Lambda'}}) \end{array} \right\}$ under $pr_i : \mathcal{X}_{\Lambda'}^n \to \mathcal{X}_{\Lambda'}$

$\to$ pull-backs of $R^*(M_{\Lambda'})$ via $\pi_{\Lambda'}^n$

<u>Fact</u> $\{R^*(\mathcal{X}_\Lambda^n)\}_{n \geq 0}$ closed under pullbacks & pushforw. via projections $\mathcal{X}_\Lambda^n \longrightarrow \mathcal{X}_\Lambda^m$

For the evaluation morphism $\varepsilon^n : \overline{M}_{g,n}(\pi_\Lambda, L) \longrightarrow \mathcal{X}_\Lambda^n$:

<u>Conjecture 1</u> $\varepsilon_*^n [\overline{M}_{g,n}(\pi_\Lambda, L)]^{red} \in R^*(\mathcal{X}_\Lambda^n)$.

For fixed K3 surface X:

<u>Conjecture 2</u> $\varepsilon^n_* [\overline{\mathcal{M}}_{g,n}(X,L)]^{red} \in A^*(X^n)$
lies in the Beauville–Voisin ring of X^n,
generated by diagonals & pullbacks of $\mathrm{Pic}(X)$.

<u>Buelles</u> $L = H$ prim. polariz. $\Rightarrow$ proves Conj. 2 in cohomol.

<u>§4 Exporting WDVV & Getzler's relation</u>

Recall the diagram

$$\begin{tikzcd} & \overline{\mathcal{M}}_{g,n}(\pi_\Lambda, L) \arrow[dl, "\tau"'] \arrow[dr, "\varepsilon^n"] & \\ \overline{M}_{g,n} & & \mathcal{X}^n_\Lambda \end{tikzcd}$$

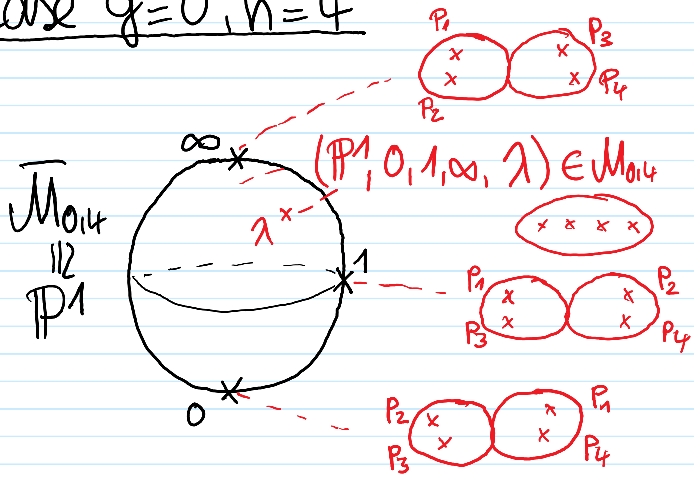
and the export operation

$$\varepsilon^n_*(\tau^* \mathcal{R}el) \cap [\overline{\mathcal{M}}_{g,n}(\pi_\Lambda, L)]^{red} = 0 \in A_*(\mathcal{X}^n_\Lambda)$$

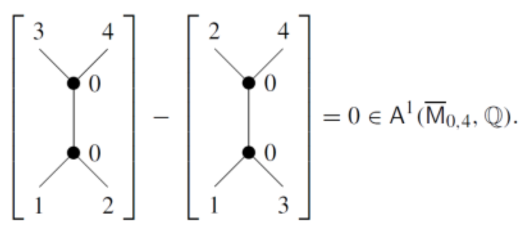
for $\mathcal{R}el = \sum R_i = 0 \in A^*(\overline{\mathcal{M}}_{g,n})$.

Pandharipande–Yin make this explicit in two cases:

<u>Case $g=0, n=4$</u>



<u>WDVV-relation</u>



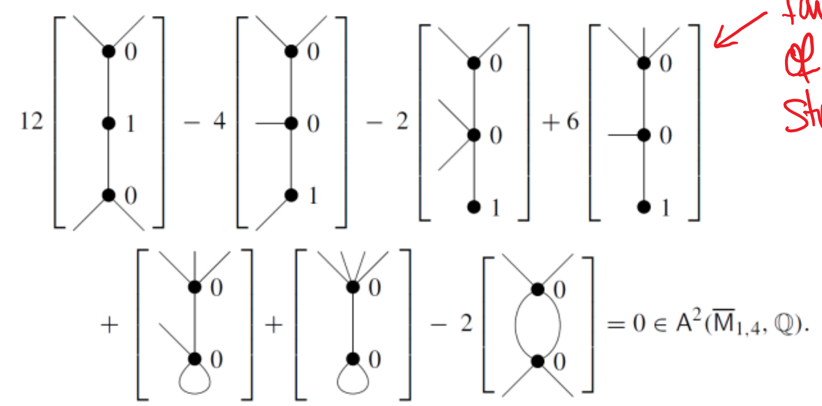
<u>Theorem 2</u> For $L \in \Lambda$ admissible, exportation of the WDVV relation yields:

taut. classes on $\mathcal{X}^4_\Lambda$ supp. over strict NL loci of M_Λ

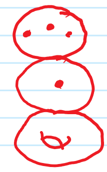
$$\mathcal{L}_{(1)}\mathcal{L}_{(2)}\mathcal{L}_{(3)}\Delta_{(34)} + \mathcal{L}_{(1)}\mathcal{L}_{(3)}\mathcal{L}_{(4)}\Delta_{(12)}$$
$$- \mathcal{L}_{(1)}\mathcal{L}_{(2)}\mathcal{L}_{(3)}\Delta_{(24)} - \mathcal{L}_{(1)}\mathcal{L}_{(2)}\mathcal{L}_{(4)}\Delta_{(13)} + \ldots = 0 \in A^5(\mathcal{X}^4_\Lambda, \mathbb{Q}), \qquad (\dagger)$$

Case $g=1, n=4$

Getzler's relation (Getzler '97, Pandh. '99)



fundam. classes of closed codim 2 strata in $\overline{M}_{1,4}$



some ordering of markings p_i

Theorem 3 For admissible $L \in \Lambda$ satisfying $L^2 \geq 0$, exportation of Getzler's relation yields

$$\mathcal{L}_{(1)}\Delta_{(12)}\Delta_{(34)} + \mathcal{L}_{(3)}\Delta_{(12)}\Delta_{(34)} + \mathcal{L}_{(1)}\Delta_{(13)}\Delta_{(24)} + \mathcal{L}_{(2)}\Delta_{(13)}\Delta_{(24)} + \mathcal{L}_{(1)}\Delta_{(14)}\Delta_{(23)}$$
$$+ \mathcal{L}_{(2)}\Delta_{(14)}\Delta_{(23)} - \mathcal{L}_{(1)}\Delta_{(234)} - \mathcal{L}_{(2)}\Delta_{(134)} - \mathcal{L}_{(3)}\Delta_{(124)} - \mathcal{L}_{(4)}\Delta_{(123)}$$
$$- \mathcal{L}_{(1)}\Delta_{(123)} - \mathcal{L}_{(1)}\Delta_{(124)} - \mathcal{L}_{(1)}\Delta_{(134)} - \mathcal{L}_{(2)}\Delta_{(234)} + \cdots = 0 \in \mathsf{A}^5(\mathcal{X}^4_\Lambda, \mathbb{Q}), \qquad (\ddagger)$$

NL terms

Obtain interesting relation in $\mathcal{X}^3_\Lambda$:
Consider projection $pr_{(123)} : \mathcal{X}^4_\Lambda \to \mathcal{X}^3_\Lambda$ to first 3 factors,
let $L = H$ q. polarization

Apply $pr_{(123)*}(\mathcal{H}_{(4)} \cdot -)$ to $(\ddagger)$.

Corollary 1 The π^3_Λ-relative diagonal $\Delta_{(123)}$ admits a decompos.

$$2\ell \cdot \Delta_{(123)} = \mathcal{H}^2_{(1)}\Delta_{(23)} + \mathcal{H}^2_{(2)}\Delta_{(13)} + \mathcal{H}^2_{(3)}\Delta_{(12)}$$
$$- \mathcal{H}^2_{(1)}\Delta_{(12)} - \mathcal{H}^2_{(1)}\Delta_{(13)} - \mathcal{H}^2_{(2)}\Delta_{(23)} + \cdots \in \mathsf{A}^4(\mathcal{X}^3_\Lambda, \mathbb{Q}), \qquad (\ddagger')$$

↳ generalizes result of Beauville-Voisin for fixed K3 X
↳ see occurence of NL terms in universal situation.

§5 Outlook on proof of Theorem 1 (MOP conj.)

How to use relations above to express K-classes as NL loci?

Exa

$$\mathcal{L}_{(1)}\mathcal{L}_{(2)}\mathcal{L}_{(3)}\Delta_{(34)} + \mathcal{L}_{(1)}\mathcal{L}_{(3)}\mathcal{L}_{(4)}\Delta_{(12)} - \mathcal{L}_{(1)}\mathcal{L}_{(2)}\mathcal{L}_{(3)}\Delta_{(24)} - \mathcal{L}_{(1)}\mathcal{L}_{(2)}\mathcal{L}_{(4)}\Delta_{(13)} + \ldots = 0 \in \mathsf{A}^5(\mathcal{X}_\Lambda^4, \mathbb{Q}), \qquad (\dagger)$$

$$(\pi_\Lambda^4)_*(- \cdot \Delta_{(12)} \cdot \Delta_{(34)})$$

$$2 \cdot \langle L, L \rangle_\Lambda \cdot K_{[L;1]} - 2 \cdot K_{[L^3;0]} + \cdots = 0 \in A^1(\mathcal{M}_\Lambda)$$

Indeed

- $\underbrace{\mathcal{L}_{(1)}\mathcal{L}_{(2)}}_{\rightsquigarrow \langle L, L\rangle}\mathcal{L}_{(3)}\underbrace{\underbrace{\Delta_{(34)}}\cdot\underbrace{\Delta_{(12)}}\cdot\underbrace{\Delta_{(34)}}}_{\Delta_{(34)}\cdot c_2(T_{\pi_\Lambda,(3)})} \xmapsto{pr_3} \langle L, L\rangle \cdot \mathcal{L} \cdot c_2(T_{\pi_\Lambda}) \xmapsto{\pi_\Lambda} \langle L, L \rangle K_{[L;1]}$

  excess int. formula

  Similar $\mathcal{L}_{(1)}\mathcal{L}_{(3)}\mathcal{L}_{(4)}\Delta_{(12)}$

- $\mathcal{L}_{(1)}\mathcal{L}_{(2)}\mathcal{L}_{(3)}\underbrace{\Delta_{(24)}\cdot\Delta_{(12)}\cdot\Delta_{(34)}}_{\Delta_{(1234)}} \xmapsto{pr_1} \mathcal{L}^3 \xmapsto{\pi_\Lambda} K_{[L^3;0]}$

  Similar $\mathcal{L}_{(1)}\mathcal{L}_{(2)}\mathcal{L}_{(4)}\Delta_{(13)}$

Upshot $\quad \langle L, L\rangle K_{[L;1]} - K_{[L^3;0]} \in NL^1(\mathcal{M}_\Lambda)$

Strategy
- Obtain further relations $\sum a_i K_{[-]} \in NL^1(\mathcal{M}_\Lambda)$
- Linear Algebra (sensitive to a_i) $\Rightarrow K_{[-]} \in NL^1(\mathcal{M}_\Lambda)$

§6 Some technical preliminaries (for next week)

Recall that for $L \in \Lambda$ admissible $\left(\begin{array}{l} L = m\tilde{L} \text{ w/ } \tilde{L} \text{ prim}, \tilde{L}^2 \geq -2 \\ H \cdot L \geq 0 \text{ and } L \text{ effect. for "="} \end{array}\right)$

we had defined

$$N_0(L) = \int_{[\overline{M}_{0,0}(X,L)]^{red}} 1 .$$

Proposition 1 $N_0(L) \neq 0$ for L admissible.

Proof Yau-Zaslow formula for $N_0(L)$ (Beauville, Bryan-Leung, Klemm-Maulik-Pandh.-Scheidegger)

L primitive, $L^2 = 2\ell \rightsquigarrow N_0(L) = n_0(\ell)$ for

$$\sum_{\ell=-1}^{\infty} n_0(\ell) \cdot q^\ell = \frac{1}{q \cdot \prod_{n=1}^{\infty}(1-q^n)^{24}} = \frac{1}{q} + 24 + 324q + 3200q^2 + \cdots$$

$\rightsquigarrow$ all coeff. positive

$L = m(L) \cdot \tilde{L}$, $\tilde{L}$ primitive :

multiple cover formula

$$N_0(L) = \sum_{r \mid m(L)} \frac{1}{r^3} \underbrace{n_0\left(\frac{L^2}{2r^2}\right)}_{>0} > 0 \qquad \square$$

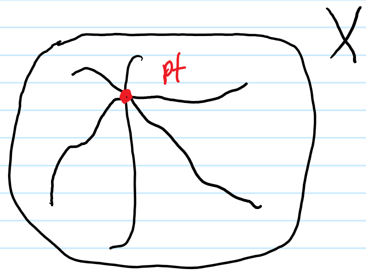
We also later need invariants in genus $g = 1$.

$$N_1(L) = \int_{[\overline{M}_{1,1}(X,L)]^{red}} ev^*([pt]) \quad , \quad ev : \overline{M}_{1,1}(X,L) \longrightarrow X$$

$$((C,p) \xrightarrow{f} X) \longmapsto f(p)$$

$[pt] \in A^2(X)$ class of pt.

<u>Intuition</u>



$L = \mathcal{O}(C)$, C genus 1
$\rightsquigarrow$ $|L|$ dim 1 lin. system
$\cong [\overline{M}_{1,0}(X,L)]^{red} \in A_1(\overline{M}_{1,0}(X,L))$

fin. many ell. curves through pt
$N_1(L)$ = virtual # of such curves

<u>Prop. 2</u> $N_1(L) \neq 0$ for L admissible, $L^2 \geq 0$.

<u>Proof</u> L primitive : explicit formula by Bryan–Leung.

L divisible : • conjectural multiple cover formula applied as above (Oberdieck–Pandharipande)

• proven by hand in case above from KKV formula (Pandharipande–Thomas) $\square$

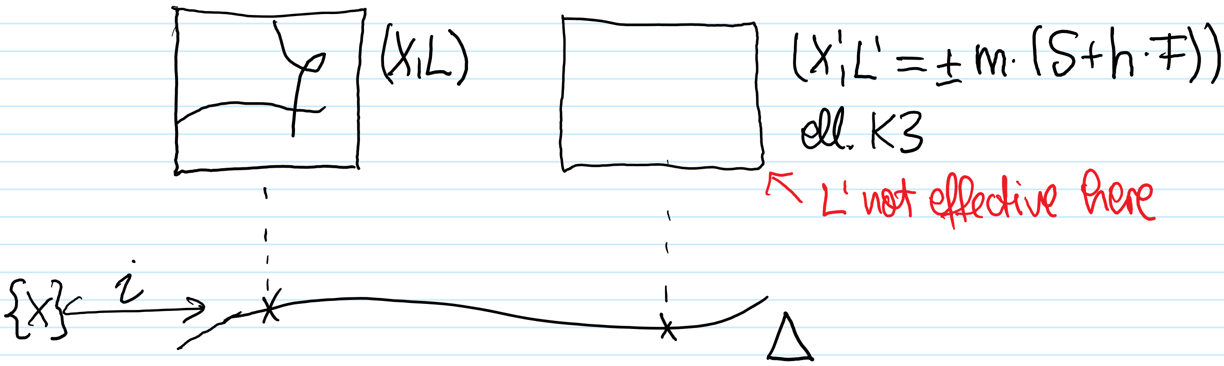
Prop. 3 L inadmissible

$\Rightarrow [\overline{M}_{g,n}(X,L)]^{red} = 0 \in A_{g+n}(\overline{M}_{g,n}(X,L), \mathbb{Q})$

Pf If $H\cdot L<0$ or $H\cdot L = 0$ & L not effective

$\Rightarrow \overline{M}_{g,n}(X,L) = \emptyset$ ✓

Otherw. assume $L = m\tilde{L}$, $\tilde{L}^2 = 2h-2 < -2$ $(\Leftrightarrow h<0)$



$$[\overline{M}_{g,n}(\pi_\Delta, L)]^{red} = 0$$

property of red. virt. class in families

$$\downarrow i^!$$

$$[\overline{M}_{g,n}(X,L)]^{red}$$

since not supported over gen. pt. of Δ

$\square$